Cauchy Eular Homogeneous Linear Differential Equations

$$a_n x^n \frac{d^n y}{d x^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{d x^{n-1}} + ... + a_0\, y = \phi(x)$$

Where $a_0,\, a_1,\, a_2,\, ...$ are constants, is called a homogeneous equation.

Put $\quad x = e^z, \quad z = \log_e\, x, \quad \frac{d}{dz} = D$

$$\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} \;\Rightarrow\; x\frac{dy}{dx} = \frac{dy}{dz} \;\Rightarrow\; x\frac{dy}{dx} = Dy$$

Again, $\quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\frac{dz}{dx}$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\frac{1}{x} = \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) = \frac{1}{x^2}(D^2 - D)y$$

$$x^2\frac{d^2y}{dx^2} = (D^2 - D)y$$

$\Rightarrow \quad \boxed{x^2\frac{d^2y}{dx^2} = D(D-1)y}$

Similarly. $\quad \boxed{x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)\,y}$

Q.1 Solve : $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = \cos(\log x) + x\sin(\log x)$.

Sol. We have, $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = \cos(\log x) + x\sin(\log x)$. …(1)

Putting $x = e^z \Rightarrow z = \log x, D = \frac{d}{dz}$ and $x^2\frac{d^2y}{dx^2} = D(D-1)y, x\frac{dy}{dx} = Dy$ in (1), we get

$$[D(D-1) - D + 4]y = \cos z + e^z \sin z$$

i.e. $(D^2-2D+4)y=\cos z+e^z\sin z$

A.E. is $m^2-2m+4=0 \Rightarrow \quad m=\frac{-(-2)\pm\sqrt{4-16}}{2}$

$\Rightarrow \quad m=1\pm\sqrt{3}i$

$\therefore$ C.F. $=e^z[C_1\cos\sqrt{3}z+C_2\sin\sqrt{3}z]$ …(2)

$$\text{P.I.}=\frac{1}{D^2-2D+4}(\cos z+e^z\sin z)$$

$$=\frac{1}{D^2-2D+4}\cos z+\frac{1}{D^2-2D+4}e^z\sin z$$

$$=\frac{1}{-1-2D+4}\cos z+e^z\frac{1}{(D+1)^2-2(D+1)+4}\sin z$$

$$=\frac{1}{3-2D}\cos z+e^z\frac{1}{D^2+2D+1-2D-2+4}\sin z$$

$$=\frac{3+2D}{9-4D^2}\cos z+e^z\frac{1}{D^2+3}\sin z=\frac{3+2D}{9+4}\cos z+e^z\frac{1}{-1+3}\sin z$$

$$=\frac{3+2D}{13}\cos z+e^z\frac{1}{2}\sin z=\frac{1}{2}\sin z=\frac{1}{13}(3\cos z-2\sin z)+\frac{1}{2}e^z\sin z \quad \ldots(3)$$

Complete solution is

$$y=C.F.+P.I.$$

$$\Rightarrow \quad y=e^z[C_1\cos\sqrt{3}z+C_2\sin\sqrt{3}z]+\frac{1}{13}(3\cos z-2\sin z)+\frac{1}{2}e^z\sin z \quad \ldots(4)$$

Replacing $z=\log x$ and $e^z=x$ in (4), we get

$$y=x[C_1\cos\sqrt{3}(\log x)+C_2\sin\sqrt{3}(\log x)]$$

$$+\frac{3}{13}\cos(\log x)-\frac{2}{13}\sin(\log x)+\frac{1}{2}x\sin(\log x)$$ Ans

Legendre's Homogeneous differential Equations

A linear differential equation of the form

$$(a+bx)^n \frac{d^n y}{dx^n} + a_1 (a+bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + ... + a_n y = X \qquad ...(1)$$

Where $a, b, a_1, a_2, ... a_n$ are constants and X is a function of x, is called Legendre's linear equation.

Put $a + b\ x = e^z \quad \Rightarrow \quad z = \log(a + bx)$ ...(1)

$$(a+bx)\frac{dy}{dx} = b\ Dy \qquad ...(2)$$

$$(a+bx)^2 \frac{d^2 y}{dx^2} = b^2\ D(D-1) \qquad ...(3)$$

$$(a+bx)^3 \frac{d^3 y}{dx^3} = b^3\ D(D-1)(D-2)y \qquad ...(4)$$

Q.2 Solve $(2x+1)^2 \frac{d^2 y}{dx^2} - 2(2x+1)\frac{dy}{dx} - 12y = 6x.$

Sol. Here, we have

$$(2x+1)^2 \frac{d^2 y}{dx^2} - 2(2x+1)\frac{dy}{dx} - 12y = 6x \qquad ...(1)$$

Put $2x+1 = e^z \quad \Rightarrow \quad z = \log(2x+1)$

$(2x+1)\frac{dy}{dx} = 2\,Dy$ and $(2x+1)^2 \frac{d^2 y}{dx^2} = 2^2 D(D-1)y$ in (1), we get

$$4D(D-1)y - 2\times 2Dy - 12y = 6\left[\frac{1}{2}(e^z - 1)\right]$$

$$4D^2 y - 4\ Dy - 4\ Dy - 12y = 3(e^z - 1)$$

A.E. is $4m^2 - 8m - 12 = 0 \quad \Rightarrow \quad m^2 - 2m - 3 = 0$

$\Rightarrow \quad (m-3)(m+1)=0 \quad \Rightarrow \quad m=3,\ m=-1$

$$C.F.=C_1e^{3z}+C_2e^{-z}$$

$$P.I.=\frac{1}{4D^2-8D-12}(3e^z-3)$$

$$=\frac{1}{4D^2-8D-12}3e^z-3\frac{1}{4D^2-8D-12}e^{(0)z}$$

$$=3\frac{1}{4(1)^2-8(1)-12}e^z-3\frac{1}{0-0-12}(1)$$

$$=\frac{3}{-16}e^z+\frac{1}{4}$$

Complete Solution is $y=C.F.+P.I.$

$$\Rightarrow \quad y=C_1e^{3z}+C_2e^{-z}-\frac{3}{16}e^z+\frac{1}{4}$$

$$\Rightarrow \quad y=C_1(2x+1)^3+C_2(2x+1)^{-1}-\frac{3}{16}(2x+1)+\frac{1}{4}$$

Ans.