Method of reduction of order:

Method 1. To Find the Complete Solution of $y'' + Py' + Qy = R$ when part of Complementary Function is known (Method of reduction of order)

Let $y = u$ be a part of the complementary function of the given differential equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad \ldots(1)$$

Where u is a function of x. Then, we have

$$\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0 \qquad \ldots(2)$$

Let $y = uv$ be the complete solution of equation (1), where v is a function of x.

Differentiating y w.r.t. x,

$$\frac{dy}{dx} = u\frac{dv}{dx} + \frac{du}{dx}.v$$

Again, $$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}.\frac{du}{dx} + v\frac{d^2u}{dx^2}$$

Substituting the values of $y, \frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in equation (1) we get

$$u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2} + P\left(u\frac{dv}{dx} + v\frac{du}{dx}\right) + Q(uv) = R$$

$$\Rightarrow \quad u\frac{d^2v}{dx^2} + \left(2\frac{du}{dx} + Pu\right)\frac{dv}{dx} + \left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right)v = R$$

$$\Rightarrow \quad u\frac{d^2v}{dx^2} + \left(2\frac{du}{dx} + Pu\right)\frac{dv}{dx} = R \qquad \text{Using (2)}$$

$$\Rightarrow \quad \frac{d^2v}{dx^2} + \left(\frac{2}{u}\frac{du}{dx} + P\right)\frac{dv}{dx} = \frac{R}{u} \qquad \ldots(3)$$

Put $\frac{dv}{dx} = P$ then, $\frac{d^2v}{dx^2} = \frac{dp}{dx}$

Now (3) becomes, $$\frac{dp}{dx} + \left(\frac{2}{u}\frac{du}{dx} + P\right)P = \frac{R}{u} \qquad \text{....(4)}$$

Equation (4) is a linear differential equation of I order in p and x.

$$I.F. = e^{\int\left(\frac{2}{u}\frac{du}{dx}+p\right)dx} = e^{\left(\int\frac{2}{u}du+\int P\,dx\right)} = u^2\, e^{\int P\,dx}$$

Solution of (4) is given by

$$pu^2\, e^{\int P\,dx} = \int \frac{R}{u}u^2\, e^{\int P\,dx}dx + c_1$$

Where c_1 is an arbitrary constant of integration.

$$\Rightarrow \qquad p = \frac{1}{u^2}e^{-\int P\,dx}\left[\int Ru\, e^{\int P\,dx}dx + c_1\right]$$

$$\therefore \qquad \frac{dv}{dx} = \frac{1}{u^2}e^{-\int P\,dx}\left[\int Ru\, e^{\int P\,dx}dx + c_1\right]$$

Integration yields, $$v = \int \frac{1}{u^2}e^{-\int P\,dx}\left[\int Ru\, e^{\int P\,dx}dx + c_1\right]dx + c_2$$

Where c_2 is an arbitrary constant of integration.

Hence the complete solution of (1) is given by,

$$y = uv$$

$$\Rightarrow \qquad y = u\int \frac{1}{u^2}e^{-\int p\,dx}\left[\int Ru\ e^{\int p\,dx}dx + c_1\right]dx + c_2 u$$

To find out the part of C.F. of the linear differential equation of II order given by

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R.$$

Remember :

	Condition	Part of C.F.
(i)	$1+\frac{P}{a}+\frac{Q}{a^2}=0$	e^{ax}
(ii)	$1+P+Q=0$	e^{x}
(iii)	$1-P+Q=0$	e^{-x}
(iv)	$m(m-1)+Pmx+Qx^2=0$	x^m
(v)	$P+Qx=0$	x
(vi)	$2+2Px+Qx^2=0$	x^2

Q.1 Solve : $\frac{d^2y}{dx^2}-\cot x\frac{dy}{dx}-(1-\cot x)y=e^x\sin x.$

Sol. Comparing with the standard form, we get

$$P=-\cot x, Q=-(1-\cot x), R=e^x\sin x$$

$$1+P+Q=1-1+\cot x-\cot x=0$$

$\therefore$ A part of $C.F.=e^x$

Let $y=ve^x$ be the complete solution of given equation, then

$$\frac{dy}{dx}=ve^x+e^x\frac{dv}{dx}$$

$$\frac{d^2y}{dx^2}=ve^x+2e^x\frac{dv}{dx}+e^x\frac{d^2v}{dx^2}$$

Substituting for $y, \frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in given equation, we get

$$\frac{d^2v}{dx^2}+(2-\cot x)\frac{dv}{dx}=\sin x$$

$\Rightarrow \qquad \dfrac{dp}{dx}+(2-\cot x)\,p=\sin x \qquad \ldots(1)$ where $p=\dfrac{dv}{dx}$.

This is a linear differential equation of I order in p and x.

$$I.F.=e^{\int(2-\cot x)dx}=\frac{e^{2x}}{\sin x}$$

Solution of (1) is, $p\dfrac{e^{2x}}{\sin x}=\displaystyle\int \sin x.\frac{e^{2x}}{\sin x}\,dx+c_1=\frac{e^{2x}}{2}+c_1$

Where c_1 is an arbitrary constant of integration.

$$p=\frac{1}{2}\sin x+c_1e^{-2x}\sin x$$

$$\frac{dv}{dx}=\frac{1}{2}\sin x+c_1e^{-2x}\sin x$$

Integrating, we get $v=-\dfrac{1}{2}\cos x-\dfrac{1}{5}c_1e^{-2x}(\cos x+2\sin x)+c_2$

Hence the complete solution is given by,

$$y=v\,e^{x}=\left[-\frac{1}{2}\cos x-\frac{1}{5}c_1e^{-2x}(\cos x+2\sin x)+c_2\right]e^{x}.$$

Reduced to Normal Form (Removal of first derivative)

Method 2. To Find the Complete Solution of $y''+Py'+Qy=R$ when it is Reduced to Normal Form (Removal of first derivative)

When the part of C.F. can not be determined by the previous method, we reduce the given differential equation in **normal form** by eliminating the term in which there exists first derivative of the dependent variable.

$$\frac{d^2y}{dx^2}P\frac{dy}{dx}+Qy=R \qquad \ldots(1)$$

Let $y=u\,v$ be the complete solution of eqn. (1), where u and v are the function of x.

$\therefore \qquad \dfrac{dy}{dx}=v\dfrac{du}{dx}+u\dfrac{dv}{dx}$

and $$\frac{d^2y}{dx^2} = v\frac{d^2u}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2}$$

Substituting the value of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in eqn. (1), we get

$$\frac{d^2v}{dx^2} + \left(\frac{2}{u}\frac{du}{dx} + P\right)\frac{du}{dx} + v\left(\frac{1}{u}\frac{d^2u}{dx^2} + \frac{P}{u}\frac{du}{dx} + Q\right) = \frac{R}{u} \quad \ldots(2)$$

Let us choose u such that $\frac{2}{u}\frac{du}{dx} + P = 0$ …(3)

Which on solving gives,

$$u = e^{-\int \frac{P}{2}dx} \quad \ldots(4)$$

From (3), $\frac{du}{dx} = -\frac{Pu}{2}$

Differentiating, we get $\frac{d^2u}{dx^2} = -\frac{1}{2}\left[P\left(\frac{du}{dx}\right) + \frac{dP}{dx}(u)\right]$

$$= -\frac{1}{2}\left[P\left(\frac{-Pu}{2}\right) + u\frac{dP}{dx}\right] = \frac{P^2u}{2} - \frac{u}{2}\frac{dP}{dx}$$

Coefficient of $v = \frac{1}{u}\frac{d^2u}{dx^2} + \frac{P}{u}\frac{du}{dx} + Q = \frac{1}{u}\left[\frac{P^2u}{4} - \frac{u}{2}\frac{dP}{dx}\right] + \frac{P}{u}\left(\frac{-Pu}{2}\right) + Q$

$$= Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4} = I \text{ (say)}$$

Then (2) becomes, $$\frac{d^2v}{dx^2} + Iv = S \quad \ldots(5)$$

This is known as the normal form of equation (1).

Solving (5), we get v in terms of x. Ultimately, $y = u\upsilon$ is the complete solution.

Q.2 Solve : $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 1)y = -3e^{x^2}\sin 2x.$

Sol. Here, $P = -4x, Q = 4x^2 - 1, R = -3e^{x^2}\sin 2x$

Let $y = u\upsilon$ be the complete solution.

Now, $$u = e^{-\frac{1}{2}\int(-4x)dx} = e^{x^2}$$

$$I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4} = 4x^2 - 1\frac{1}{2}(-4) - \frac{1}{4}(16x^2) = 1.$$

Also, $$S = \frac{R}{u} = \frac{-3e^{x^2}\sin 2x}{e^{x^2}} = -3\sin 2x$$

Hence normal form is,

$$\frac{d^2\upsilon}{dx^2} + \upsilon = -3\sin 2x$$

Auxiliary equation is

$$m^2 + 1 = 0 \quad \Rightarrow \quad m = \pm\, i$$

$$C.F. = c_1\cos x + c_2\sin x$$

where c_1 and c_2 are arbitrary constants of integration.

$$P.I. = \frac{1}{D^2+1}(-3\sin 2x) = \frac{-3}{(-4+1)}\sin 2x = \sin 2x$$

$\therefore$ Solution is, $\upsilon = c_1\cos x + c_2\sin x + \sin 2x$

Hence the complete solution of given differential equation is

$$y = u\upsilon = e^{x^2}(c_1\cos x + c_2\sin x + \sin 2x).$$