Matrix Theory

Various types of Matrices

(a) **Row Matrix**-. If a matrix has only one row and any number of columns, it is called a Row matrix, e.g.,

[2 7 3 9]

(a) **Column Matrix**.- A matrix, having one column and any number of rows, is called a Column matrix, e.g., $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(b) **Null Matrix**- or Zero Matrix. Any matrix, in which all the elements are zeros, is called a Zero matrix or Null matrix e.g.,

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(c) **Square Matrix**. A matrix, in which the number of rows is equal to the number of columns, is called a square matrix e.g.,

$$\begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}$$

(d) **Diagonal Matrix**. A square matrix is called a diagonal matrix, if all its non-diagonal elements are zero e.g.,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

(e) **Scalar Matrix**. A diagonal matrix in which all the diagonal elements are equal to a scalar, say (k) is called a scalar matrix.
For example;

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} -6 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & -6 \end{bmatrix}$$

i.e., $A = [a_{ij}]_{n \times n}$ is a scalar matrix if $[a_{ij}] = \begin{cases} 0, & wnen\, i \neq j \\ k, & when\, i = j \end{cases}$

(g) **Unit or Identity Matrix**. A square matrix is called a unit matrix if all the diagonal elements are unity and non-diagonal elements are zero e.g.,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(f) **Symmetric Matrix**. A square matrix will be called symmetric, if for all values of I and j, $a_{ij} = a_{ij}$ i.e., $A' = A$

$$\text{e.g., } \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

(i) **Skew Symmetric Matrix**. A square matrix is called skew symmetric matrix, if

(1) $a_{ij} = -a_{ij}$ for all values of i and j, or $A' = -A$

(2) All diagonal elements are zero, e.g.,

$$\begin{bmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{bmatrix}$$

(j) **Triangular Matrix**. (Echelon form) A square matrix, all of whose elements below the leading diagonal are zero, is called an *upper triangular matrix*. A square matrix, all of whose elements above the leading diagonal are zero, is called *a lower triangular matrix* e.g.,

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & 6 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & 6 & 7 \end{bmatrix}$$

Upper triangular matrix Lower triangular matrix

(k) **Transpose of a Matrix**. If in a given matrix A, we interchange the rows and the corresponding columns, the new matrix obtained is called the transpose of the matrix A and is denoted by A' or A^T e.g.,

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 5 \\ 6 & 7 & 8 \end{bmatrix}, \; A' = \begin{bmatrix} 2 & 1 & 6 \\ 3 & 0 & 7 \\ 4 & 5 & 8 \end{bmatrix}$$

(l) Matrix A^{θ}. Transpose of the conjugate of a matrix A is denoted by A^{θ}.

Let $$A = \begin{bmatrix} 1+i & 2-3i & 4 \\ 7+2i & -i & 3-2i \end{bmatrix}$$

$$\overline{A} = \begin{bmatrix} 1+i & 2+3i & 4 \\ 7+2i & +i & 3+2i \end{bmatrix}$$

$$(\overline{A})' = \begin{bmatrix} 1-i & 7-2i \\ 2+3i & i \\ 4 & 3+2i \end{bmatrix}$$

$$A^{\theta} = \begin{bmatrix} 1-i & 7-2i \\ 2+3i & i \\ 4 & 3+2i \end{bmatrix}$$

(m) **Skew Hermitian Matrix**. A square matrix $A = (a_{ij})$ will be called a Skew Hermitian matrix if every i-jth element of A is equal to negative conjugate complex j-ith element of A.

In other words, $a_{ij} = -\bar{a}_{ji}$

All the elements in the principal diagonal will be of the form

$a_{ii} = -\bar{a}_{ii}$ or $a_{ii} = -\bar{a}_{ii} = 0$

if $a_{ii} = a + ib$ then $\bar{a}_{ii} = a - ib$

$(a+ib)+(a-ib)=0 \Rightarrow 2a = 0 \Rightarrow a = 0$ $[a_{ii} + \bar{a}_{ii} + 0]$

So, a_{ii} is pure imaginary $\Rightarrow a_{ii} = 0 + ib = ib$

Hence, all the diagonal elements of a Skew Hermitian Matrix are either zeros or pure imaginary.

e.g. $$\begin{bmatrix} i & 2-3i & 4+5i \\ -(2+3i) & 0 & 2i \\ -(4-5i) & 2i & -3i \end{bmatrix}$$

The necessary and sufficient condition for a matrix A to be Skew Hermitian is that

$$A^{\theta} = -A$$

$$(\bar{A})' = -A$$

(n) **Idempotent Matrix**. A matrix, such that $A^2 = A$ is called Idempotent Matrix.

e.g. $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$, $A^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$

(O) **Periodic Matrix**. A matrix A will be called a Periodic Matrix, if

$$A^{k+1} = A$$

Where k is a +ve integer. If k is the least +ve integer, for which $A^{k+1} = A$, then k is said to be the period of A. if we choose $k = 1$, we get $A^2 = A$ and we call it to be idempotent matrix.

(p) **Nilpotent matrix**. A matrix will be called a Nilpotent matrix, if $A^k = 0$ (null matrix) where k is a +ve integer; if however k is the least +ve integer for which $A^k = 0$, then k is the index of the nilpotent matrix.

e.g., $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} A^2 = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}\begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

A is nilpotent matrix whose index is 2.

(q) **Involuntary Matrix**. A matrix A will be called an Involuntary matrix, if $A^2 = I$ (unit matrix). Since $I^2 = I$ always $\therefore$ Unit matrix is involuntary.

(r) **Equal Matrices**. Two matrices are said to be equal if
(i) They are of the same order.
(ii) The elements in the corresponding positions are equal.

Thus if $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

Here $A = B$

(s) **Singular Matrix**. If the determinate of the matrix is zero, then the matrix is known as singular matrix e.g. $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ is singular matrix, because $|A| = 6 - 6 = 0$.

The inverse of a symmetric Matrix

The elementary transformations are to be transformed so that the property of being symmetric is preserved. This requires that the transformations occur in pairs, a row transformation must be followed immediately by the same column transformation.

Ques. Find the inverse of the following matrix employing elementary transformations:

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

Sol. The given matrix is $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad \Rightarrow \begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 \to \frac{R_1}{3} \\ A \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 0 & -1 & \frac{4}{3} \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} A \\ R_2 \to R_2 - 2R_1 \end{matrix} \Rightarrow \begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 0 & 1 & \frac{4}{3} \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} A \\ R_2 \to -R_2 \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & -1 & 0 \\ \frac{2}{3} & -1 & 1 \end{bmatrix} A \qquad R_3 \to R_3 + R_2$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & -1 & 0 \\ -2 & 3 & -3 \end{bmatrix} A \qquad R_3 \to -3R_3 + R_2$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 4 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} A \qquad R_1 \to R_1 - \frac{4}{3}R_3, \quad R_2 \to R_2 + \frac{4}{3}R_3$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} A \qquad R_1 \to R_1 + R_2$$

Hence,
$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

Ans.

Rank of Matrix (Echelon & Normal Form)

Rank of a Matrix-

The rank of a matrix is said to be r if

(a) It has at least one non-zero minor of order r.

(b) Every minor of A of order higher than r is zero.

Note: (i) Non-zero row is that row in which all the elements are not zero.

(ii) The rank of the product matrix AB of two matrices A and B is less than the rank of either of the matrices A and B.

(iii) Corresponding to every matrix A of rank r, there exist non-singular matrices P and Q such that $PAQ = \begin{bmatrix} Ir & 0 \\ 0 & 0 \end{bmatrix}$

Normal Form (Canonical Form)

By performing elementary transformation, any non-zero matrix A can be reduced to one of the following four forms, called the Normal form of A:

(i) I (ii) $[I_r \quad 0]$ (iii) $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ (iv) $\begin{bmatrix} Ir & 0 \\ 0 & 0 \end{bmatrix}$

The number r so obtained is called the rank of A and we write $\rho(A) = r$. The form $\begin{bmatrix} Ir & 0 \\ 0 & 0 \end{bmatrix}$ is called first canonical form of A. Since both row and column transformations may be used here, the element 1 of the first row obtained can be moved in the first column. Then both the first row and first column can be cleared of other non-zero elements. Similarly, the element 1 of the second row can be brought into the second column, and so on.

Ques. Find the rank of the matrix .

$$A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ -3 & 2 & 4 & 5 \end{bmatrix}$$

Sol. Here, we have

$$A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ -3 & 2 & 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 11 & 19 & 26 \end{bmatrix} R_2 \to R_2 + 3R_1$$

The number on non-zero row is 2, therefore Rank $(A) = 2$

Exp. 10 If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, Find two non singular matrices P and Q such that $PAQ = 1$. Hence find A^{-1}.

Sol. $A_{3\times 3} = I_3 \, A I_3$

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0\\ 2 & -3 & 4\\ 0 & -1 & 1\end{bmatrix}=\begin{bmatrix}1 & -1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}A\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}R_1\rightarrow R_1-R_2$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & -3 & 4\\ 0 & -1 & 1\end{bmatrix}=\begin{bmatrix}1 & -1 & 0\\ -2 & 3 & 0\\ 0 & 0 & 1\end{bmatrix}A\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}R_3\rightarrow R_2-2R_1$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 3 & 4\\ 0 & 1 & 1\end{bmatrix}=\begin{bmatrix}1 & -1 & 0\\ -2 & 3 & 0\\ 0 & 0 & 1\end{bmatrix}A\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}C_2\rightarrow -C_2$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 3 & 4\end{bmatrix}=\begin{bmatrix}1 & -1 & 0\\ 0 & 0 & 1\\ -2 & 3 & 0\end{bmatrix}A\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}R_2\leftrightarrow R_3$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & -1 & 0\\ 0 & 0 & 1\\ -2 & 3 & -3\end{bmatrix}A\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}R_3\rightarrow R_3-3R_2$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & -1 & 0\\ 0 & 0 & 1\\ -2 & 3 & -3\end{bmatrix}A\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 1\\ 0 & 0 & 1\end{bmatrix}C_3\rightarrow C_3-C_2$$

$$I_3=PAQ$$

$A^{-1}=QP,$

$$A^{-1}=\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & -1 & 0\\ 0 & 0 & 1\\ -2 & 3 & -3\end{bmatrix}\qquad \left[\begin{array}{l}I=PAQ\\ P^{-1}=AQ\\ P^{-1}Q^{-1}=A\\ (P^{-1}Q^{-1})^{-1}=A^{-1}\\ QP=A^{-1}\end{array}\right.$$

$$\Rightarrow\qquad A^{-1}=\begin{bmatrix}1 & -1 & 0\\ -2 & 3 & -4\\ -2 & 3 & -3\end{bmatrix}$$

Ans.

<u>Rank of matrix by triangular form</u>

Rank = Number of non-zero row in upper triangular matrix.

Note. Non-zero row is that row which does not contain all the elements as zero.

Exp.11 Find the rank of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & I \\ I & 3 & 4 & 5 \end{bmatrix}$$

Sol.

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & I \\ I & 3 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - R_2$$

Rank $=$ Number of non zero rows $= 2$. Ans.

Vectors

A n-tuple is a set of n similar things. If the place of every members of a set is fixed then it is called an ordered set. Any ordered n-tuple of numbers is called a n-vector. Thus the coordinates of a point in space is called 3-vector $(x,\ y,\ z)$. The members of a set are called the components of a vector so $x,\ y,\ z$ in a 3-vector are called components.

$x_1,\ x_2,\ x_3....x_n$ are the components of a n-vector $X = (x_1,\ x_2,\ x_3....x_n)$.

Each row of a matrix is a vector and each column of the matrix is also a vector.

Linear Dependence and Independence of Vectors

Vectors (matrices) $X_1, X_2,....X_n$ are said to be dependent if

(1) all the vectors (row or column matrices) are of the same order.

(2) n scalars $\lambda_1, \lambda_2,...\lambda_n$ (not all zero) exist such that

$$\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 + ... + \lambda_n X_n = 0$$

Otherwise they are linearly independent.

Ques Define linear dependence and independence of vectors.

Examine for linear dependence $[1,\ 0,\ 2,\ 1], [3,\ 1,\ 2,\ 1], [4,\ 6,\ 2,\ -4], [-6,\ 0,\ -3,\ -4]$ and find the relation between them, if possible.

Sol. Consider the matrix equation

$$\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 + \lambda_4 X_4 = 0 \qquad ...(1)$$

$$\lambda_1(1,\ 0,\ 2,\ 1) + \lambda_2(3,\ 1,\ 2,\ 1) + \lambda_3(4,\ 6,\ 2,\ -4) + \lambda_4(-6,\ 0,\ -3,\ -4) = 0$$

$$\lambda_1 + 3\lambda_2 + 4\lambda_3 - 6\lambda_4 = 0$$
$$0\lambda_1 + \lambda_2 + 6\lambda_3 + 0\lambda_4 = 0$$
$$2\lambda_1 + 2\lambda_2 + 2\lambda_3 - 3\lambda_4 = 0$$
$$\lambda_1 + \lambda_2 - 4\lambda_3 - 4\lambda_4 = 0$$

$$\begin{bmatrix} 1 & 3 & 4 & -6 \\ 0 & 1 & 6 & 0 \\ 2 & 2 & 2 & -3 \\ 1 & 1 & -4 & -4 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 4 & -6 \\ 0 & 1 & 6 & 0 \\ 0 & -4 & -6 & 9 \\ 0 & -2 & -8 & 2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} \\ \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 3 & 4 & -6 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 18 & 9 \\ 0 & 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} \\ \\ R_3 \to R_3 + 4R_2 \\ R_4 \to R_4 + 2R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 3 & 4 & -6 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 18 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} R_4 \to R_4 - \frac{2}{9} R_3$$

$$\lambda_1 + 3\lambda_2 + 4\lambda_3 - 6\lambda_4 = 0$$
$$\lambda_2 + 6\lambda_3 = 0$$
$$18\lambda_3 + 9\lambda_4 = 0$$

Let $\lambda_4 = t,$ $\quad 18\lambda_3 + 9t = 0$ or $\lambda_3 = \frac{-t}{2}$

$\lambda_2 - 3t = 0$ or $\lambda_2 - 3t$

$\lambda_1 + 9t - 2t - 6t = 0$

$\lambda_1 = -t$

Substituting the values of $\lambda_1, \lambda_2, \lambda_3$ and λ_4 in (1), we get

$-t\,X_1 + 3t\,X_2 - \frac{t}{2}X_3 + t\,X_4 = 0$ or $2X_t - 6X_2 + X_3 - 2X_4 = 0$ Ans.

Solution of Simultaneous Equations

The matrix of the coefficients of x, y, z is reduced into Echelon form by elementary row transformations. At the end of the row transformation the value of z is calculated from the last equation and value of y and the value of x are calculated by the backward substitution.

Exp.9 Test the consistency and hence solve the following set of equation.

$$x_1 + 2x_2 + x_3 = 2$$
$$3x_1 + x_2 - 2x_3 = 1$$
$$4x_1 - 3x_2 - x_3 = 3$$
$$2x_1 + 4x_2 + 2x_3 = 4$$

Sol. The given set of equations is written in the matrix form:

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & -2 \\ 4 & -3 & -1 \\ 2 & 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}$$

$$AX = B$$

Here, we have augmented matrix $C = [A, B] \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 1 & -2 & 1 \\ 4 & -3 & -1 & 3 \\ 2 & 4 & 2 & 4 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -5 & -5 & -5 \\ 0 & -11 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} \\ R_2 \to R_2 - 3\,R_1 \\ R_3 \to R_3 - 4\,R_1 \\ R_4 \to R_4 - 2\,R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & -11 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 \to -\frac{1}{5}\,R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 6 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \rightarrow R_3 + 11\, R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \rightarrow \frac{1}{6}\, R_3$$

Number of non-zero rows = Rank of matrix.

$$\Rightarrow \qquad R(C) = R(A) = 3$$

Hence, the given system is consistent and possesses and unique solution. In matrix form the system reduces to

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 + x_3 = 2$$

$$x_2 + x_3 = 1$$

$$x_3 = 1$$

From (2), $x_2 + 1 = 1 \Rightarrow x_2 = 0$

From (1) $x_1 + 0 + 1 = 2 \Rightarrow x_1 = 1$

Hence, $x_1 = 1,\ x_2 = 0$ and $x_3 = 1$

Types of Linear Equations

(1)<u>Consistent.</u> A system of equations is said to be consistent, if they have one or more solution i.e.

$$x + 2y = 4 \qquad\qquad x + 2y = 4$$

$$3x + 2y = 2 \qquad\qquad 3x + 6y = 12$$

Unique solution — Infinite solution

(2) <u>Inconsistent.</u> If a system of equation has no solution, it is said to be inconsistent i.e.

$$x + 2y = 4$$

$$3x + 6y = 5$$

<u>Consistency of A System of Linear Equations</u>

$$a_{11}x_1 + a_{12}x_2 + \ldots a_{In}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots a_{2n}x_n = b_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{22}x_2 + \ldots a_{2n}x_n = b_m$$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12}\ldots\ldots\ldots a_{1n} \\ a_{21} & a_{22}\ldots\ldots\ldots a_{2n} \\ \ldots\ldots & \ldots\ldots\ldots\ldots \\ a_{m1} & a_{m2}\ldots\ldots\ldots a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_m \end{bmatrix}$$

$$\Rightarrow \qquad AX = B$$

and

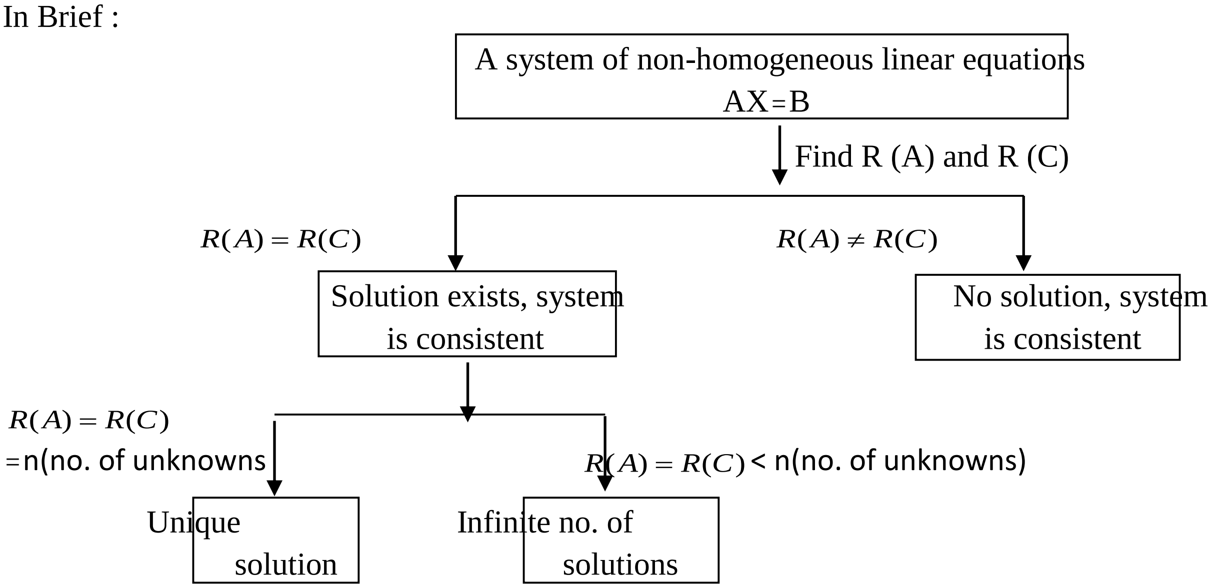
$$C = [A, B] = \begin{bmatrix} a_{11} & a_{12}\ldots\ldots a_{1n} & b_1 \\ a_{21} & a_{22}\ldots\ldots a_{2n} & b_2 \\ \ldots\ldots & \ldots\ldots\ldots\ldots & \ldots \\ a_{m1} & a_{m2}\ldots\ldots a_{mn} & b_m \end{bmatrix}$$

is called the augmented matrix. $[A:B] = C$

(a) Consistent equations. If Rank A = Rank C
(i) Unique solution : Rank A = Rank C = n
(ii) Infinite solution: Rank A = Rank C = r, r < n

(b) Inconsistent equations. If Rank A $\neq$ Rank C.

In Brief :



Exp.12 Show that the equations

$2x+6y=-11,\ 6x+20y-6z=-3,\ 6y-18z=-1$

Are not consistent.

Sol. Augmented matrix $C=[A,\ B]$

$$=\begin{bmatrix}2 & 6 & 0 & : & -11\\ 6 & 20 & -6 & : & -3\\ 0 & 6 & -18 & : & -1\end{bmatrix}\sim\begin{bmatrix}2 & 6 & 0 & : & -11\\ 0 & 2 & -6 & : & 30\\ 0 & 6 & -18 & : & -1\end{bmatrix} R_2\to R_2-3R_1$$

$$\sim\begin{bmatrix}2 & 6 & 0 & : & -11\\ 0 & 2 & -6 & : & 30\\ 0 & 0 & 0 & : & -91\end{bmatrix} R_3\to R_3-3R_2$$

The rank of C is 3 and the rank of A is 2

Rank of A $\neq$ Rank of C.

The equations are not consistent.

Ques. Investigate the values of λ and μ so that the equations:

$$2x+3y+5z=9$$
$$7x+3y-2z=8$$
$$2x+3y+\lambda z=\mu$$

Have (i) no solution

(ii) a unique solution

(iii) an infinite number of solutions.

Sol. Here, we have,

$$2x+3y+5z=9$$
$$7x+3y-2z=8$$
$$2x+3y+\lambda z=\mu$$

The above equations are written in the matrix form

$$\begin{bmatrix}2 & 3 & 5\\ 7 & 3 & -2\\ 2 & 3 & \lambda\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}9\\ 8\\ \mu\end{bmatrix}$$

$$AX=B$$

$$C=[A:B]=\begin{bmatrix}2 & 3 & 5 & : & 9\\ 7 & 3 & -2 & : & 8\\ 2 & 3 & \lambda & : & \mu\end{bmatrix}$$

$$=\begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 0 & -\frac{15}{2} & -\frac{39}{2} & : & -\frac{47}{2} \\ 0 & 0 & \lambda-5 & : & \mu-9 \end{bmatrix} \begin{matrix} R_2 \to R_2 - \frac{7}{2}R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

(i) No solution.

Rank $(A) \neq$ Rank (C)

$\lambda - 5 \neq 0$ or $\lambda = 5$ and $\mu - 9 \neq 0$ $\Rightarrow \mu \neq 9$

(ii) A unique solution

Rank $(A) = R(C) =$ Number of unknowns

$\lambda - 5 \neq 0$ $\Rightarrow$ $\lambda \neq 5$

(iii) An infinite number of solutions.

Rank $(A) =$ Rank $(C) = 2$

$\lambda - 5 = 0$ and $\mu - 9 = 0$

$\lambda = 5$ and $\mu = 9$ Ans.

Homogeneous Equations

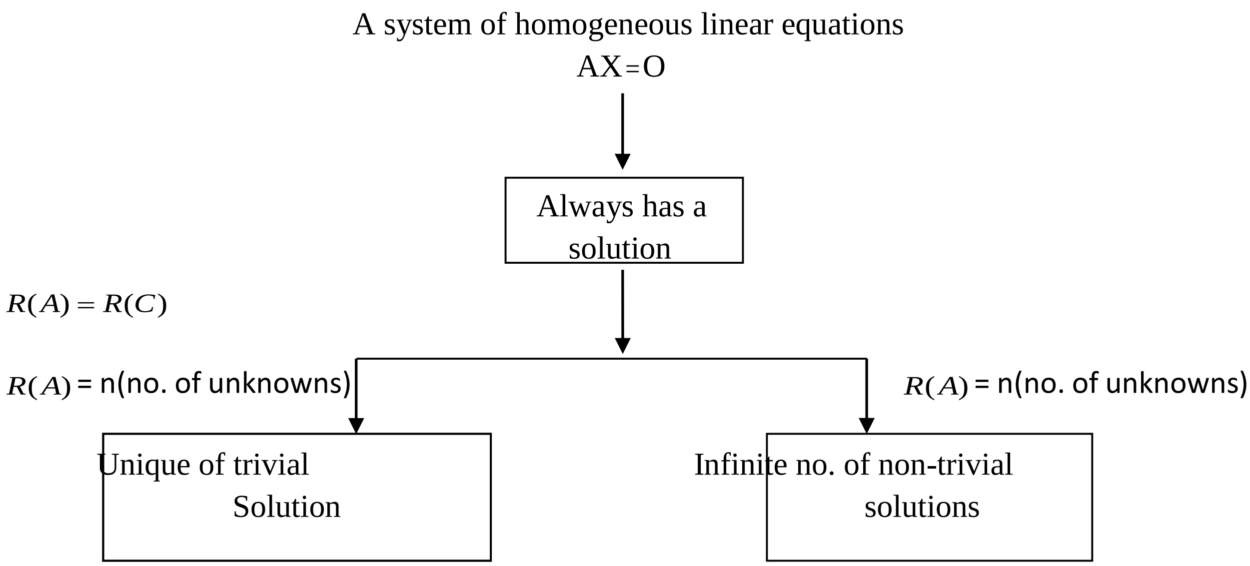
For a system of homogeneous linear equations $AX = O$

(i) $X = O$ is always a solution. This solution in which each unknown has the value zero is called the Null Solution or the Trivial solution. Thus a homogeneous system is always consistent.

A system of homogeneous linear equations has either the trivial solution or an infinite number of solution.

(ii) If $R(A) =$ number of unknowns, the system has only the trivial solution.

(iii) If $R(A) <$ number of unknowns, the system has an infinite number of non- trivial solution.



(each unknown equal to zero)

Exp.25 Determine 'b' such that the system of homogeneous equations

$$2x + y + 2z = 0;$$
$$x + y + 3z = 0;$$
$$4x + 3y + bz = 0$$

has (i) Trivial solution

(ii) Non-Trivial solution. Find the Non-Trivial solution using matrix method.

(U.P., I Sem Dec 2008)

Sol. Here, we have

$$2x + y + 2z = 0$$
$$x + y + 3z = 0$$
$$4x + 3y + bz = 0$$

(i) For trivial solution: We know that $x = 0,\ y = 0$ and $z = 0.$ So, b can have any value.

(ii) For non-trivial solution: The given equation are written in the matrix form as:

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & 3 & b \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$AX = B$$

$$R_1 \leftrightarrow R_2, \quad R_2 \to R_2 - 2R_1, R_3 \to R_3 - 4R_1, \quad R_3 \to R_3 - R_2$$

$$C = \begin{bmatrix} 2 & 1 & 2 & 0 \\ 1 & 1 & 3 & 0 \\ 4 & 3 & b & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 3 & b & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & -1 & b-12 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & b-8 & 0 \end{bmatrix}$$

For non trivial solution or infinite solutions $RCC = R(A) = 2 <$ Number of unknowns

$b - 8 = 0, \quad b = 8$ Ans.

Eigen values

Eigen values and Eigen vectors are used in the study of ordinary differential equations, analysing population growth and finding powers of matrices.

Eigen Values

Let $$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}$$

$$AX = Y$$

Where A is the matrix, X is the column vector and Y is also column vector. …(1)

Here column vector X is transformed into the column vector Y by means of the square matrix A.

Let X be a such vector which transforms into λX by means of the transformation (1). Suppose the linear transformation $Y = AX$ transforms X into a scalar multiple of itself i.e. λX .

$$AX = Y = \lambda X$$

$$AX - \lambda IX = 0 \qquad \text{…(2)}$$

$$(A - \lambda I)X = 0$$

Thus the unknown scalar λ is known as an Eigen value of the matrix A and the corresponding non zero vector X as Eigen vector.

The Eigen values are also called characteristic values or proper values or latent values.

Let $$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}$$ Characteristic matrix

Characteristic Polynomial: The determinant $|A - \lambda I|$ when expanded will give a polynomial, which we call as characteristic polynomial of matrix A.

Some Important Properties of Eigen Values

(1) Any square matrix A and its transpose A' have the same Eigen values.

Note. The sum of the elements on the principle diagonal of a matrix is called the trace of the matrix.

(2) The sum of the Eigen values of a matrix is equal to the trace of the matrix.

(3) The product of the Eigen values of a matrix A is equal to the determinant of A.

(4) If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the Eigen values of A, then the Eigen values of

(i) k A are $k\lambda_1, \; k\lambda_2, \; \ldots \; k\lambda_n$ (ii) A^m are $\lambda_1^m, \lambda_2^m, \ldots\ldots\lambda_n^m$

(iii) A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$.

Cayley-Hamilton Theorem

Satement. Every square matrix satisfies its own characteristic equation.

If $|A-\lambda I| = (-1)^n(\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n)$ be the characteristic polynomial of $n\times n$ matrix $A=(a_{ij})$, then the matrix equation

$X^n + a_1X^{n-1} + a_2X^{n-2} + \ldots + a_nI = 0$ is satisfied by $X = A$ i.e.,

$$A^n + a_1A^{n-1} + a_2A^{n-2} + \ldots + a_nI = 0$$

Exp.18 Verify Cayley- Hamilton theorem for the matrix

$A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ and hence find A^{-1}.

Sol. The characteristic equation of the matrix is

$$|A-\lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(-1-\lambda) - 4 = 0 \Rightarrow -1 + \lambda^2 - 4 = 0 \Rightarrow \lambda^2 - 5 = 0$$

By Cayley-Hamilton Theorem,

$$A^2 - 5I = 0 \qquad \ldots(1)$$

Now, $A^2 = A.A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}$

$$A^2 - 5I = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} + \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \qquad \ldots(2)$$

From (1) and (2), Cayley-Hamilton theorem is verified.

Again from (1), we have

$$A^2 - 5I = 0$$

Multiplying by A^{-1}, we get

$$A - 5A^{-1} = 0 \Rightarrow A^{-1} = \frac{1}{5}A$$

$$A^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix} \qquad \text{Ans}$$

Exp.19 Find the characteristic equation of the matrix

$$A=\begin{bmatrix}1 & 2 & -2\\ 1 & 1 & 1\\ 1 & 3 & -1\end{bmatrix}$$

Hence find A^{-1}.

Sol. Characteristic equation of matrix A is

$$|A-\lambda I|=0$$

$$\begin{vmatrix}1-\lambda & 2 & -2\\ 1 & 1-\lambda & 1\\ 1 & 3 & -1-\lambda\end{vmatrix}=0$$

$\Rightarrow$ $(1-\lambda)[(1-\lambda)(-1-\lambda)-3]-2(-1-\lambda-1)-2(3-1+\lambda)=0$

$\Rightarrow$ $(1-\lambda)(-1+\lambda^2-3)-2(-\lambda-2)-2(2+\lambda)=0$

$\Rightarrow$ $(1-\lambda)(\lambda^2-4)+2\lambda+4-4-2\lambda=0$

$\Rightarrow$ $(-\lambda+1)(\lambda^2-4)=0$ or $\lambda^3-\lambda^2-4\lambda+4=0$

By Cayley-Hamilton Theorem

$$A^3-A^2-4A+4I=0$$

$\Rightarrow$ $A^2-A-4I+4A^{-1}=0$ (Multiplying by A^{-1})

$\Rightarrow$ $4A^{-1}=[-A^2+A+4I]$ …(1)

Now $A^2=\begin{bmatrix}1 & 2 & -2\\ 1 & 1 & 1\\ 1 & 3 & -1\end{bmatrix}\begin{bmatrix}1 & 2 & -2\\ 1 & 1 & 1\\ 1 & 3 & -1\end{bmatrix}=\begin{bmatrix}1 & -2 & 2\\ 3 & 6 & -2\\ 3 & 2 & 2\end{bmatrix}$

From (1), we have

$$4A^{-1}=-\begin{bmatrix}1 & -2 & 2\\ 3 & 6 & -2\\ 3 & 2 & 2\end{bmatrix}+\begin{bmatrix}1 & -2 & 2\\ 3 & 6 & -2\\ 3 & 2 & 2\end{bmatrix}+\begin{bmatrix}4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{bmatrix}$$

$$=\begin{bmatrix}1-+1+4 & 2+2+0 & -2-2+0\\ -3+1+0 & -6+1+4 & 2+1+0\\ -3+1+0 & -2+3+0 & -2-1+4\end{bmatrix}$$

$$A^{-1}=\frac{1}{4}\begin{bmatrix}4 & 4 & -4\\ -2 & -1 & 3\\ -2 & 1 & 1\end{bmatrix}$$

Characteristic vectors or Eigen vectors

As we have discussed in Art 3.16,

A column vector X is transformed into column vector Y by means of a square matrix A.

Now we want to multiply the column vector X by a scalar quantity λ so that we can find the same transformed column vector Y.

i.e., $AX = \lambda X$

X is known as Eigen vector.

Properties of Eigen vectors

1. The Eigen vector X of a matrix A is not unique.
2. If $\lambda_1, \lambda_2, ..., \lambda_n$ be distinct Eigen values of an $n \times n$ matrix then corresponding Eigen vectors $X_1, X_2, ..., X_n$ form a linearly independent set.
3. If two or more Eigen values are equal it may or may not be possible to get linearly independent Eigen vectors corresponding to the equal roots.
4. Two Eigen vectors x_1 and x_2 are called orthogonal vectors if $X_1' X_2 = 0.$
5. Eigen vectors of a symmetric matrix corresponding to different Eigen values are orthogonal.

Ques . Find the Eigen value and corresponding Eigen vectors of the matrix

$$A = \begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix}$$

Sol. $|A - \lambda I| = 0$

$$\Rightarrow \quad \begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0 \Rightarrow (-5-\lambda) - 4 = 0$$

$$\Rightarrow \quad \lambda^2 + 7\lambda + 10 - 4 = 0 \Rightarrow \lambda^2 + 7\lambda + 6 = 0$$

$$(\lambda + 1)(\lambda + 6) = 0 \Rightarrow \lambda = -1, -6$$

The Eigen values of the given matrix are -1 and -6.

(i) When $\lambda = -1$, the corresponding Eigen vectors are given by

$$\begin{bmatrix} -5+1 & 2 \\ 2 & -2+1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad 2x_1 - x_2 = 0 \Rightarrow x_1 = \frac{1}{2}x_2$$

Let $x_1 = k$, then $x_2 = 2k$

Hence, Eigen vector $X_1 = \begin{bmatrix} k \\ 2k \end{bmatrix}$

(i) When $\lambda = -6$, the corresponding Eigen vectors are given by

$$\begin{bmatrix} -5+6 & 2 \\ 2 & -2+6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x_1 + 2x_2 = 0 \qquad \Rightarrow x_1 = -2x_2$$

Let $x_1 = k_1$, then $x_2 = -\frac{1}{2}k_1$

Hence Eigen vector $X_2 = \begin{bmatrix} k_1 \\ -\frac{k_1}{2} \end{bmatrix}$ or $\begin{bmatrix} 2k_1 \\ -k_1 \end{bmatrix}$

Hence Eigen vectors are $\begin{bmatrix} k \\ 2k \end{bmatrix}$ and $\begin{bmatrix} 2k_1 \\ -k_1 \end{bmatrix}$

Ques . Show that the matrix $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ has less than three linearly independent Eigen vectors. Is it possible to obtain a similarity transformation that will diagonalise this matrix?

Sol. $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{vmatrix} = 0 \qquad \Rightarrow \qquad \lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$$

By trial

Let $\lambda = 2$, then $8 - 28 + 32 - 12 = 0$. $\qquad \therefore\ (\lambda - 2)$ is one factor

$$\begin{array}{c|cccc} 2 & 1 & -7 & 16 & -12 \\ & & 2 & -10 & 12 \\ \hline & 1 & -5 & 6 & 0 \end{array}$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 2)(\lambda 2 - 5\lambda + 6) = 0 \Rightarrow (\lambda - 2)(\lambda - 2)(\lambda - 3) = 0, \lambda = 2, 2, 3$$

Eigen vector for $\lambda = 3$

$$\begin{vmatrix} 3-3 & 10 & 5 \\ -2 & -3-3 & -4 \\ 3 & 5 & 7-3 \end{vmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 10 & 5 \\ -2 & -6 & -4 \\ 3 & 5 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{l} 0x + 10y + 5z = 0 \\ -2x - 6y - 4z = 0 \end{array} \qquad \Rightarrow \frac{x}{-40+30} = \frac{y}{-10-0} = \frac{z}{0+20} \Rightarrow \frac{x}{1} = \frac{y}{1} = \frac{z}{-2}$$

Eigen vector $= \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

Eigen vector for $\lambda = 2$

$$\begin{bmatrix} 1 & 10 & 5 \\ -2 & -5 & -4 \\ 3 & 5 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{i.e.,} \begin{bmatrix} x + 10y + 5z = 0 \\ 2x + 5y + 4z = 0 \end{bmatrix} \Rightarrow \frac{x}{40-25} = \frac{y}{10-4} = \frac{z}{5-20}$$

$\Rightarrow \quad \frac{x}{5} = \frac{y}{2} = \frac{z}{-5} = k$, Eigen vector $= \begin{bmatrix} 5k \\ 2k \\ -5k \end{bmatrix}$

We get one Eigen vector corresponding to repeated root $\lambda_2 = 2 = \lambda_3$.

Eigen vector corresponding to $\lambda_2 = 2 = \lambda_3$ are not linearly independent.

Similarity transformation is not possible.

Ques. Find the Eigen values, Eigen vectors the modal matrix and diagonalise the matrix given below.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

Sol. The characteristic equation of the given matrix is

$$\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{bmatrix} = 0$$

$\Rightarrow \quad (1-\lambda)\{(3-\lambda)^2 - 1\} = 0 \quad \Rightarrow \quad (1-\lambda)(3-\lambda+1)(3-\lambda-1) = 0$

$\Rightarrow \quad (1-\lambda)(4-\lambda)(2-\lambda) = 0 \quad \Rightarrow \quad \lambda = 1, 2, 4$

Eigen vectors

When $\lambda = 1$,

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & \frac{3}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} R_3 \to R_3 + \frac{1}{2}R_2$$

$\Rightarrow \quad 2x_2 - x_3 = 0$ ...(1)

$\frac{3}{2}x_3 = 0 \Rightarrow x_3 = 0$ ...(2)

Putting $x_3 = 0$ from (2) in (1), we get $2x_2 - 0 = 0 \Rightarrow x_2 = 0$

Eigen Vector $= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

When $\lambda = 2$,

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} R_1 \to -R_1 \\ R_3 \to R_3 + R_2 \end{matrix}$$

$x_1 = 0$

$x_2 - x_3 = 0 \Rightarrow x_2 = x_3$

Eigen vector $= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

When $\lambda = 4$,

$$\begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-3x_1 = 0$

$-x_2 - x_3 = 0$

$x_2 = -x_3$

Eigen Vector $= \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

Model matrix $= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

Let us diagonalise the given matrix:

$$P^{-1}AP = -\frac{1}{2}\begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 2 & -4 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Diagonalization of a matrix

Theorem. If a square matrix A of order n has n linearly independent Eigen vectors, then a matrix P can be found such that $P^{-1}\ AP$ is a diagonal matrix.

Ques.Find a matrix P which diagonalizes the matrix

$A=\begin{bmatrix}4 & 1\\ 2 & 3\end{bmatrix}$, verify $P^{-1}\ AP=D$ where D is the diagonal matrix.

Sol. The characteristic equation of matrix A is

$$\begin{vmatrix}4-\lambda & 1\\ 2 & 3-\lambda\end{vmatrix}=0 \quad\Rightarrow\quad (4-\lambda)\ (3-\lambda)-2=0$$

$$\Rightarrow \lambda^2-7\lambda+12-2=0 \quad\Rightarrow\quad \lambda^2-7\lambda+10=0$$

$$\Rightarrow (\lambda-2)(\lambda-5)=0 \quad\Rightarrow\quad \lambda=2,\lambda=5$$

Eigen values are 2 and 5.

(i) When $\lambda=2$, Eigen vectors are given by the matrix equation

$$\begin{bmatrix}4-2 & 1\\ 2 & 3-2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}\Rightarrow\begin{bmatrix}2 & 1\\ 2 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$\Rightarrow$ $2x_1+x_2=0\Rightarrow x_2=-2x_1$

Let $x_1=k,\ x_2=-2k$

Hence, the Eigen vector $X_1=\begin{bmatrix}k\\ -2k\end{bmatrix}$ or $\begin{bmatrix}1\\ -2\end{bmatrix}$

(ii) When $\lambda=5$, Eigen vectors are given by the matrix equation

$$\begin{bmatrix}4-5 & 1\\ 2 & 3-5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}\Rightarrow\begin{bmatrix}-1 & 1\\ 2 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$\Rightarrow$ $-x_1+x_2=0\Rightarrow x_1=x_2$

Let $x_1=k$, then $x_2=k$

Hence, the Eigen vector $X_2=\begin{bmatrix}k\\ k\end{bmatrix}$ or $\begin{bmatrix}1\\ 1\end{bmatrix}$

Modal matrix $P=\begin{bmatrix}1 & 1\\ -2 & 1\end{bmatrix}$ $\Rightarrow P^{-1}=\frac{1}{3}\begin{bmatrix}1 & -1\\ 2 & 1\end{bmatrix}$

For diagonalization

$$D=P^{-1}AP=\frac{1}{3}\begin{bmatrix}1 & -1\\ 2 & 1\end{bmatrix}\begin{bmatrix}4 & 1\\ 2 & 3\end{bmatrix}\begin{bmatrix}1 & 1\\ -2 & 1\end{bmatrix}$$

$$=\frac{1}{3}\begin{bmatrix}1 & -1\\ 2 & 1\end{bmatrix}\begin{bmatrix}2 & 5\\ -4 & 5\end{bmatrix}=\frac{1}{3}\begin{bmatrix}6 & 0\\ 0 & 15\end{bmatrix}=\begin{bmatrix}2 & 0\\ 0 & 5\end{bmatrix}$$ Verified.

Assignment-III (Unit-III)

1. Reduce the following matrices in to normal form and find the rank**.**

(a) $\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 0 & 1 & 2 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

2. Reduce the following matrices in to echelon form and find the rank**.**

.(a) $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 3 & -2 & 4 \\ 3 & -2 & 1 & 2 \\ 3 & 2 & 3 & 4 \\ -2 & 4 & 0 & 5 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -3 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 3 & 4 & 1 & -2 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & 4 & 1 & 1 \\ 2 & 4 & 3 & 6 \\ -1 & -2 & 6 & 4 \\ 1 & -1 & 2 & -3 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 2 \\ 1 & 2 & 3 & 2 \end{bmatrix}$

3.Find the inverse of the matrix by applying elementary transformations.

(a) $\begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & -1 & -1 \\ 2 & 1 & 2 & 1 \\ 3 & -2 & 1 & 6 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 3 & 2 \\ 2 & 4 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

(4) Determine b, such that the system of equations $2x+y+2z=0$; $x+y+3z=0$; $4x+3y+bz=0$ has

(i)trivial solution (ii) Non trivial solution. Find the non trivial solution using matrix method

(5) Test the consistency of following system of linear equations and hence find the solution

$4x_1 - x_2 = 12, \quad -x_1 + 5x_2 - 2x_3 = 0, \quad -2x_2 + 4x_3 = -8$

(6) Using matrix method, show that equations $3x + 3y + 2z = 1$, $x + 2y = 4$, $10y + 3z = -2$ and $2x - 3y - z = 5$ are consistent and hence obtain the solution for x, y and z.

(7) Investigate for what values of a, b the following equations

$x + 2y + 3z = 4, \quad x + 3y + 4z = 5 \quad$ and $\quad x + 3y + az = b$

have i) no solution ii) a unique solution iii) an infinite number of solutions.

(8) find the value of λ such that the following equations have unique solution:

$\lambda x - 2y - 2z - 1 = 0, \quad 4x + 2\lambda y - z - 2 = 0 \quad$ and $\quad 6x + 6y + \lambda z - 3 = 0$

and use matrix method to solve these equations, when $\lambda = 2$.

(9) For what values of λ and μ the following equations:

$$x + 2y + 3z = 10, x + y + z = 6, \quad x + 2y + \lambda z = \mu$$

have i) unique solution ii) no solution iii) infinite number of solutions

(10) Find the values of k for which the equations,

$(3k – 8) x + 3y + 3z = 0$, $3x + (3k – 8)y + 3z = 0$, $3x + 3y + (3k – 8)z = 0$ have a non-trivial solutions.

(11) Find values of λ for which the following system of equations is consistent and has non-trivial sols.. Solve equatios for all such values of λ :

$(\lambda - 1)x + (3\lambda + 1)y + 2\lambda z = 0$, $(\lambda - 1)x + (4\lambda - 2)y + (\lambda + 3)z = 0$, $2x + (3\lambda + 1)y + 3(\lambda-1)z = 0$

(12) For what values of k, the equations, $x + y + z = 1$, $2x + y + 4z = k$, $4x + y + 10z = k^2$ have a sol. and solve them completely in each case. (Ans- k = 1, 2)

(13) Discuss the consistency of the following equations and if consistent, find the solution:

a) $x + y + z + 3 = 0$, $3x + y – 2z + 2 = 0$, $2x + 4y + 7z – 7 = 0$.

b) $5x +3 y + 7z = 4$,$3x + 26y +2z = 9$, $7x + 2y + 10z = 5$.

(14) Find the Eigen values and Eigen vectors of the following matrices:

i) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ ii) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ iii) $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ iv) $\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(15) Examine the linear dependence of the vectors $X_1 = (2,2,1),\ X_2 = (1,3,1),\ X_3 = (1,2,2)$.

(16) Verify Cayley-Hamilton theorem for A and hence find A^{-1}, when $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

(17) State and prove Cayley- Hamilton's theorem.

(18) Use Cayley- Hamilton theorem to find out the inverse of the following matrix: $\begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$.