Ex. 4. *(a) Using Picard's method of successive approximation, find a sequence of two functions which approach solution of the initial value problem* $dy/dx = e^x + y^2, y(0) = 1$.

[Delhi Maths (Hons.) 1994, 2002]

Sol. Given problem is $\quad dy/dx = e^x + y^2, \quad$ where $\quad y = 1 \quad$ when $\quad x = 0.$ ...(1)

We know that the nth approximation y_n of the initial value problem

$dy/dx = f(x, y), \quad$ where $\quad y = y_0 \quad$ when $\quad x = x_0$ ...(2)

is given by
$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx. \qquad ...(3)$$

Comparing (1) and (2), $\quad f(x, y) = e^x + y^2, \quad x_0 = 0 \quad$ and $\quad y_0 = 1.$ ...(4)

$\therefore$ from (3),
$$y_n = 1 + \int_0^x (e^x + y_{n-1}^2)\, dx \qquad ...(5)$$

First Approximation. Putting $n = 1$ in (5), and using (4), we get

$$y_1 = 1 + \int_0^x (e^x + y_0^2)\, dx = 1 + \int_0^x (e^x + 1)\, dx = 1 + \left[e^x + x\right]_0^x = 1 + e^x + x - 1 = e^x + x. \qquad ...(6)$$

Second Approximation. Putting $n = 2$ in (5), we have

$$y_2 = 1 + \int_0^x (e^x + y_1^2)\, dx = 1 + \int_0^x \left[e^x + (e^x + x)^2\right] dx, \text{ by (6)}$$

$$= 1 + \int_0^x (e^x + e^{2x} + x^2 + 2x\, e^x)\, dx = 1 + \left[e^x + \frac{1}{2}e^{2x} + \frac{x^3}{3}\right]_0^x + 2\int_0^x x\, e^x\, dx$$

$$= 1 + e^x + \frac{1}{2}e^{2x} + \frac{x^3}{3} - 1 - \frac{1}{2} + 2\left\{\left[x\, e^x\right]_0^x - \int_0^x (1 \cdot e^x)\, dx\right\} = e^x + \frac{1}{2}e^{2x} + \frac{1}{3}x^3 - \frac{1}{2} + 2\left[xe^x - [e^x]_0^x\right]$$

$$= e^x + (1/2) \times e^{2x} + x^3/3 - (1/2) + 2x\, e^x - 2\,(e^x - 1) = (1/2) \times e^{2x} + x^3/3 + 3/2 + (2x - 1)\, e^x.$$

Ex. 4(b). *Find three successive approximations of the solution of* $dy/dx = e^x + y^2, y(0) = 0$.

[Delhi Maths (Hons.) 2007]

Sol. Given problem in $\quad dy/dx = e^x + y^2, \quad y(0) = 0$ ... (1)

We know that the nth approximation y_n of the initial value problem

$dy/dx = f(x, y), \quad$ where $\quad y = y_0 \quad$ when $\quad x = x_0$ ... (2)

in given by
$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx \qquad ... (3)$$

Comparing (1) and (2), here $\quad f(x, y) = e^x + y^2, \quad x_0 = 0, \quad y_0 = 0$ ... (4)

$\therefore$ (3) reduces to
$$y_n = \int_0^x (e^x + y_{n-1}^2)\, dx \qquad ... (5)$$

First approximation . Putting $n = 1$ in (5) and using (4), we get

$$y_1 = \int_0^x (e^x + y_0^2)\, dx = \int_0^x e^x\, dx = [e^x]_0^x = e^x - 1 \qquad (6)$$

Second approximation : Putting $n = 2$ in (5) and using (4) and (6), we get

$$y_2 = \int_0^x (e^x + y_1^2)\, dx = \int_0^x \{e^x + (e^x - 1)^2\}\, dx = \int_0^x (e^{2x} - e^x + 1)\, dx$$

$$= [(1/2) \times e^{2x} - e^x + x]_0^x = (1/2) \times (e^{2x} - 2e^x + 2x + 1) \qquad ... (7)$$

Third approximation : Putting $n = 3$ in (5) and using (4) and (7), we get

$$y_3 = \int_0^x (e^x + y_2^2)\,dx = \int_0^x \{e^x + (1/4)\times(e^{2x} - 2e^x + 2x + 1)^2\}\,dx$$

$$= \frac{1}{4}\int_0^x (4e^x + e^{4x} - 4e^{2x} + 4x^2 + 1 - 4e^{3x} + 4xe^{2x} + 2e^{2x} - 8xe^x - 4e^x + 4x)\,dx$$

$$= \frac{1}{4}\int_0^x (e^{4x} - 4e^{3x} + 2e^{2x} + 4x^2 + 4x + 1 + 4xe^{2x} - 8xe^x)\,dx$$

$$= \frac{1}{16}\left[e^{4x}\right]_0^x - \frac{1}{3}\left[e^{3x}\right]_0^x + \frac{1}{4}\left[e^{2x}\right]_0^x + \frac{1}{4}\left[\frac{4x^3}{3} + 2x^2 + x\right]_0^x + \left[x\times\left(\frac{1}{2}e^{2x}\right) - (1)\times\left(\frac{1}{4}e^{2x}\right)\right]_0^x$$

$$-2\left[(x)(e^x) - (1)\times(e^x)\right]_0^x, \text{ on integrating by parts the last two terms}$$

$$= (1/16)\times(e^{4x} - 1) - (1/3)\times(e^{3x} - 1) + (1/4)\times(e^{2x} - 1) + (1/12)\times(4x^3 - 6x^2 + 3x)$$

$$+ (x/2)\times e^{2x} - (1/4)\times e^{2x} + 1/4 - 2(xe^x - e^x + 1)$$

$$= (1/16)\times e^{4x} - (1/3)\times e^{3x} + (1/4)\times e^{2x} + (1/12)\times(4x^3 - 6x^2 + 3x) + (1/4)\times(2x - 1)e^{2x} - 2(x - 1)e^x - (83/48).$$

Ex. 5. *Use Picard's method to obtain a solution of the differential equation.* $dy/dx = x^2 - y$, $y(0) = 0$. *Find at least the fourth approximation to each solution.* **[Meerut 1996]**

Sol. Given $dy/dx = x^2 - y$, where $y = 0$ when $x = 0$. ...(1)

We know that the *n*th approximation y_n of the initial value problem

$dy/dx = f(x, y)$, when $y = y_0$ when $x = x_0$ ...(2)

is given by $$y_n = y_0 + \int_{x_0}^x f(x, y_{n-1})\,dx. \qquad ...(3)$$

Comparing (1) and (2), $f(x, y) = x^2 - y$, $x_0 = 0$ and $y_0 = 0$. ...(4)

$\therefore$ from (3), $$y_n = \int_0^x (x^2 - y_{n-1})\,dx. \qquad ...(5)$$

First approximation. Putting $n = 1$ in (5) and using (4), we get

$$y_1 = \int_0^x (x^2 - y_0)\,dx = \int_0^x x^2\,dx = \left[\frac{1}{3}x^3\right]_0^x = \frac{1}{3}x^3. \qquad ...(6)$$

Second approximation. Putting $n = 2$ in (5) and using (6), we get

$$y_2 = \int_0^x (x^2 - y_1)\,dx = \int_0^x \left(x^2 - \frac{1}{3}x^3\right)dx = \left[\frac{1}{3}x^3 - \frac{1}{12}x^4\right]_0^x = \frac{1}{3}x^3 - \frac{1}{12}x^4. \qquad ...(7)$$

Third approximation. Putting $n = 3$ in (5) and using (7), we get

$$y_3 = \int_0^x (x^2 - y_2)\,dx = \int_0^x \left[x^2 - \left(\frac{1}{3}x^3 - \frac{1}{12}x^4\right)\right]dx = \left[\frac{1}{3}x^3 - \frac{1}{12}x^4 + \frac{1}{60}x^5\right]_0^x$$

or $$y_3 = x^3/3 - x^4/12 + x^5/60. \qquad ...(8)$$

Fourth approximation. Putting $n = 4$ in (5) and using (7), we get

$$y_4 = \int_0^x (x^2 - y_3)\,dx = \int_0^x \left[x^2 - \left(\frac{1}{3}x^3 - \frac{1}{12}x^4 + \frac{1}{60}x^5\right)\right] dx = \left[\frac{1}{3}x^3 - \frac{1}{12}x^4 + \frac{1}{60}x^5 - \frac{1}{360}x^6\right]_0^x$$

or $y = x^3/3 - x^4/12 + x^5/60 - x^6/360.$

Ex. 6. (a) *Apply Picard's method to find the solution of the problem $dy/dx = y - x$, $y(0) = 2$. Show that the iterative solution approaches the exact solution.* **[Meerut 1995]**

Sol. Given $dy/dx = y - x$, where $y = 2$ when $x = 0$. ...(1)

We know that the nth approximation of the initial value problem

$dy/dx = f(x, y)$ where $y = y_0$ when $x = x_0$ ...(2)

is given by $$y_n = y_0 + \int_{x_0}^x f(x, y_{n-1})\,dx. \quad ...(3)$$

Comparing (1) and (2), $f(x, y) = y - x$, $x_0 = 1$ and $y_0 = 2$. ...(4)

$\therefore$ from (3), $$y_n = 2 + \int_0^x (y_{n-1} - x)\,dx. \quad ...(5)$$

First approximation. Putting $n = 1$ in (5) and using (4), we get

$$y_1 = 2 + \int_0^x (y_0 - x)\,dx = 2 + \int_0^x (2 - x)\,dx = 2 + 2x - \frac{1}{2}x^2. \quad ...(6)$$

Second approximation. Putting $n = 2$ in (5) and using (6) we get

$$y_2 = 2 + \int_0^x (y_1 - x)\,dx = 2 + \int_0^x \left(2 + 2x - \frac{1}{2}x^2 - x\right) dx = 2 + 2x + \frac{1}{2}x^2 - \frac{1}{6}x^3 \quad ...(7)$$

Third approximation. Putting n = 3 in (5) and using (7), we get

$$y_3 = 2 + \int_0^x (y_2 - x)\,dx = 2 + \int_0^x [2 + 2x + x^2/2 - x^3/6 - x]\,dx$$

$$= 2 + 2x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} = 1 + x + 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} \quad ...(8)$$

To find the exact solution of (1). Rewriting (1), we have

$(dy/dx) - y = -x$, which is a linear differential equation ...(9)

Its I.F. $= e^{\int(-1)dx} = e^{-x}$ and hence its solution is

$$ye^{-x} = \int (-x)(e^{-x})\,dx + c = -\left[x(-e^{-x}) - \int 1.(-e^{-x})\,dx\right] + c,$$ c being an arbitrary constant

or $ye^{-x} = xe^{-x} + e^{-x} + c$ or $y = x + 1 + c\,e^x$. ...(10)

Given that $y = 2$, when $x = 0$, so (10) gives $2 = 1 + c$ or $c = 1$.

Hence, from (10), the exact solution is $y = x + 1 + e^x$. ...(11)

We know that $e^x = 1 + x + (x^2/2!) + (x^3/3!) + \ldots\ldots$ ad inf. ...(12)

Keeping (12) and (8) in view, we find that the approximate solution tends to

$y = 1 + x + 1 + x/1! + x^2/2! + x^3/3! + \ldots = 1 + x + e^x$ *i.e.*, which is exact solution of (4).