Picard-Lindelöf Theorem

Theorem *The space $\mathcal{C}([a,b])$ of continuous functions from $[a,b]$ to $\mathbb{R}^n$ equipped with the norm*

$$\|f\|_\infty := \sup\{|f(x)| \mid x \in [a,b]\}$$

is a Banach space.

Definition Two different norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space $\mathcal{X}$ are *equivalent* if there exist constants $m, M > 0$ such that

$$m\|x\|_1 \le \|x\|_2 \le M\|x\|_1$$

for every $x \in \mathcal{X}$.

Theorem *If $(\mathcal{X}, \|\cdot\|_1)$ is a Banach space and $\|\cdot\|_2$ is equivalent to $\|\cdot\|_1$ on $\mathcal{X}$, then $(\mathcal{X}, \|\cdot\|_2)$ is a Banach space.*

Theorem *A closed subspace of a complete metric space is a complete metric space.*

We are now in a position to state and prove the Picard-Lindelöf Existence-Uniqueness Theorem. Recall that we are dealing with the IVP

$$\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = a. \end{cases} \tag{1}$$

Theorem (Picard-Lindelöf) *Suppose $f : [t_0 - \alpha, t_0 + \alpha] \times \overline{B(a,\beta)} \to \mathbb{R}^n$ is continuous and bounded by M. Suppose, furthermore, that $f(t,\cdot)$ is Lipschitz continuous with Lipschitz constant L for every $t \in [t_0 - \alpha, t_0 + \alpha]$. Then (1) has a unique solution defined on $[t_0 - b, t_0 + b]$, where $b = \min\{\alpha, \beta/M\}$.*

Proof. Let $\mathcal{X}$ be the set of continuous functions from $[t_0 - b, t_0 + b]$ to $\overline{B(a,\beta)}$. The norm

$$\|g\|_w := \sup\{e^{-2L|t-t_0|}|g(t)| \mid t \in [t_0 - b, t_0 + b]\}$$

is equivalent to the standard supremum norm $\|\cdot\|_\infty$ on $C([t_0 - b, t_0 + b])$, so this vector space is complete under this weighted norm. The set $\mathcal{X}$ endowed with this norm/metric is a closed subset of this complete Banach space, so $\mathcal{X}$ equipped with the metric $d(x_1, x_2) := \|x_1 - x_2\|_w$ is a complete metric space.

Given $x \in \mathcal{X}$, define $T(x)$ to be the function on $[t_0 - b, t_0 + b]$ given by the formula

$$T(x)(t) = a + \int_{t_0}^{t} f(s, x(s))\, dx.$$

Step 1: If $x \in \mathcal{X}$ then $T(x)$ makes sense.
This should be obvious.

Step 2: If $x \in \mathcal{X}$ then $T(x) \in \mathcal{X}$.
If $x \in \mathcal{X}$, then it is clear that $T(x)$ is continuous (and, in fact, differentiable). Furthermore, for $t \in [t_0 - b, t_0 + b]$

$$|T(x)(t) - a| = \left| \int_{t_0}^{t} f(s, x(s))\, ds \right| \le \left| \int_{t_0}^{t} |f(s, x(s))|\, ds \right| \le Mb \le \beta,$$

so $T(x)(t) \in \overline{B(a, \beta)}$. Hence, $T(x) \in \mathcal{X}$.

Step 3: T is a contraction on $\mathcal{X}$.
Let $x, y \in \mathcal{X}$, and note that $\|T(x) - T(y)\|_w$ is

$$\sup\left\{ e^{-2L|t-t_0|} \left| \int_{t_0}^{t} [f(s, x(s)) - f(s, y(s))]\, ds \right| \;\middle|\; t \in [t_0 - b, t_0 + b] \right\}.$$

For a fixed $t \in [t_0 - b, t_0 + b]$,

$$\begin{aligned}
& e^{-2L|t-t_0|} \left| \int_{t_0}^{t} [f(s, x(s)) - f(s, y(s))]\, ds \right| \\
& \le e^{-2L|t-t_0|} \left| \int_{t_0}^{t} |f(s, x(s)) - f(s, y(s))|\, ds \right| \\
& \le e^{-2L|t-t_0|} \left| \int_{t_0}^{t} L|x(s) - y(s)|\, ds \right| \\
& \le Le^{-2L|t-t_0|} \left| \int_{t_0}^{t} \|x - y\|_w e^{2L|s-t_0|}\, ds \right| \\
& = \frac{\|x - y\|_w}{2} \left(1 - e^{-2L|t-t_0|}\right) \\
& \le \frac{1}{2}\|x - y\|_w.
\end{aligned}$$

Taking the supremum over all $t \in [t_0 - b, t_0 + b]$, we find that T is a contraction (with $\lambda = 1/2$).

By the contraction mapping principle, we therefore know that T has a unique fixed point in $\mathcal{X}$. This means that (1) has a unique solution in $\mathcal{X}$ (which is the only place a solution could be). □