① MPHY-CC-6 Electrodynamics & plasma physics unit-1

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Electric field vector in terms of scalar and vector potentials & wave equation in terms of scalar and vector potential:

* Electric field vector in terms of scalar potential ϕ and vector potential $\vec{A}$:

Maxwell's equation of electromagnetism is

$$\vec{\nabla}\cdot\vec{D} = \rho \quad \text{——— (A)}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \text{——— (B)}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad \text{——— (C)}$$

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t} \quad \text{——— (D)}$$

From equⁿ (B), $\vec{\nabla}\cdot\vec{B} = 0$ or $\text{div}\,\vec{B} = 0$

As we know that $\text{div}(\text{curl}\,\vec{A}) = 0$ i.e, $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$

Because divergence of curl of any vector is zero

$$\Rightarrow \vec{B} = \text{curl}\,\vec{A} \quad \text{or} \quad \vec{B} = \vec{\nabla}\times\vec{A} \quad \text{——— (1)}$$

where $\vec{A}$ is vector potential.

Therefore, curl of vector potential $\vec{A}$ is equal to magnetic field $\vec{B}$.

From equⁿ (C), $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

$$\Rightarrow \vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A}) \quad \text{using equ}^n\ (1)$$

$$\Rightarrow \vec{\nabla}\times\vec{E} = -\vec{\nabla}\times\frac{\delta\vec{A}}{\delta t} \Rightarrow \vec{\nabla}\times\vec{E} + \vec{\nabla}\times\frac{\partial\vec{A}}{\partial t} = 0$$

$$\Rightarrow \vec{\nabla}\times\left(\vec{E} + \frac{\delta\vec{A}}{\delta t}\right) = 0 \quad \text{——— (2)}$$

As we know that $\vec{\nabla}\times(\vec{\nabla}\phi) = 0$ or $\vec{\nabla}\times(-\vec{\nabla}\phi) = 0$

Because curl of gradient of any scalar quantity is zero

On comparing with equⁿ (2), we get

Dr. Md. NAIYAR PERWEZ

$$-\vec{\nabla}\phi = \vec{E} + \frac{\delta \vec{A}}{\delta t}$$

where ϕ = scalar potential

$$\Rightarrow \vec{E} = -\vec{\nabla}\phi - \frac{\delta \vec{A}}{\delta t}$$

$$\Rightarrow \boxed{\vec{E} = -\left(\vec{\nabla}\phi + \frac{\delta \vec{A}}{\delta t}\right) = -\left(\text{grad}\,\phi + \frac{\delta \vec{A}}{\delta t}\right)}$$

It is formula for electric field vector $\vec{E}$ in terms of scalar potential ϕ and vector potential $\vec{A}$.

* **Derivation of wave equation in terms of scalar ~~and~~ potential ϕ and vector potential $\vec{A}$:**

Maxwell's equation of electromagnetism is

$$\vec{\nabla}\cdot\vec{D} = \rho \quad \text{——— (A)}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \text{——— (B)}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\delta \vec{B}}{\delta t} \quad \text{——— (C)}$$

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\delta \vec{D}}{\delta t} \quad \text{——— (D)}$$

From equ^n (B), $\vec{\nabla}\cdot\vec{B} = 0$ or $\text{div}\,\vec{B} = 0$

As we know that $\text{div}(\text{curl}\,\vec{A}) = 0$ or $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$

Because divergence of curl of any vector is zero

$$\Rightarrow \vec{B} = \vec{\nabla}\times\vec{A} \quad \text{or} \quad \vec{B} = \text{curl}\,\vec{A} \quad \text{——— (1)}$$

Where $\vec{A}$ = vector potential

Therefore, curl of vector potential $\vec{A}$ is equal to magnetic field vector $\vec{B}$.

From equ^n (C), $\vec{\nabla}\times\vec{E} = -\frac{\delta \vec{B}}{\delta t}$

$$\Rightarrow \vec{\nabla}\times\vec{E} = -\frac{\delta}{\delta t}(\vec{\nabla}\times\vec{A}) \quad \because \text{using equ}^n (1)$$

$$\Rightarrow \vec{\nabla}\times\vec{E} = -\vec{\nabla}\times\frac{\delta\vec{A}}{\delta t} \Rightarrow \vec{\nabla}\times\vec{E} + \vec{\nabla}\times\frac{\delta\vec{A}}{\delta t} = 0$$

$$\Rightarrow \vec{\nabla}\times\left(\vec{E}+\frac{\delta\vec{A}}{\delta t}\right) = 0 \quad \text{——— (2)}$$

As we know that $\vec{\nabla}\times(\vec{\nabla}\phi)=0$ or $\vec{\nabla}\times(-\vec{\nabla}\phi)=0$

Because curl of gradient of any scalar quantity is equal to zero

on comparing with equⁿ ②, we get

$$-\vec{\nabla}\phi = \vec{E}+\frac{\delta\vec{A}}{\delta t}$$

where ϕ = scalar potential

$$\Rightarrow \vec{E} = -\vec{\nabla}\phi - \frac{\delta\vec{A}}{\delta t}$$

$$\Rightarrow \vec{E} = -\left(\vec{\nabla}\phi + \frac{\delta\vec{A}}{dt}\right) = -\left(\text{grad}\,\phi + \frac{\delta\vec{A}}{\delta t}\right) \quad \text{——— ③}$$

From equⁿ (D), $\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\delta\vec{D}}{\delta t}$

$$\Rightarrow \vec{\nabla}\times\frac{\vec{B}}{\mu} = \vec{J} + \frac{\delta\epsilon\vec{E}}{\delta t} \qquad \because \vec{H}=\frac{\vec{B}}{\mu} \text{ and } \vec{D}=\epsilon\vec{E}$$

$$\Rightarrow \vec{\nabla}\times\vec{B} = \mu\vec{J} + \mu\varepsilon\frac{\delta\vec{E}}{\delta t} \quad \text{——— ④}$$

Using equⁿˢ ① and ③ in equⁿ ④, we get

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu\vec{J} + \mu\varepsilon\frac{\delta}{\delta t}\left(-\vec{\nabla}\phi - \frac{\delta\vec{A}}{\delta t}\right)$$

$$\Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu\vec{J} - \mu\varepsilon\frac{\delta}{\delta t}\left(\vec{\nabla}\phi + \frac{\delta\vec{A}}{\delta t}\right) \quad \text{——— (5)}$$

Now $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{A}(\vec{\nabla}\cdot\vec{\nabla})$

$$\because \vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$$

$$\Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{A}\,\nabla^2 \qquad \because \vec{\nabla}\cdot\vec{\nabla} = \nabla^2$$

$$\Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \text{grad}(\text{div}\,\vec{A}) - \nabla^2\vec{A} \quad \text{——— (6)}$$

put in equⁿ ⑤, we get

Dr. Md. NAIYAR PERWEZ

$$\text{grad}(\text{div}\vec{A}) - \nabla^2\vec{A} = \mu\vec{J} - \mu\epsilon\frac{\delta}{\delta t}\left(\text{grad}\,\phi + \frac{\delta\vec{A}}{\delta t}\right)$$

$$\Rightarrow \cancel{\text{grad}(\text{div}\vec{A}) - \nabla^2\vec{A} = \mu\vec{J} - \mu\epsilon\,\text{grad}\,\phi - \mu\epsilon}$$

$$\Rightarrow \text{grad}(\text{div}\vec{A}) - \nabla^2\vec{A} = \mu\vec{J} - \mu\epsilon\,\text{grad}\frac{\delta\phi}{\delta t} - \mu\epsilon\frac{\delta^2\vec{A}}{\delta t^2}$$

$$\Rightarrow \nabla^2\vec{A} - \mu\epsilon\frac{\delta^2\vec{A}}{\delta t^2} - \mu\epsilon\,\text{grad}\frac{\delta\phi}{\delta t} - \text{grad}(\text{div}\vec{A}) = -\mu\vec{J}$$

$$\Rightarrow \boxed{\nabla^2\vec{A} - \mu\epsilon\frac{\delta^2\vec{A}}{\delta t^2} - \text{grad}\left(\text{div}\vec{A} + \mu\epsilon\frac{\delta\phi}{\delta t}\right) = -\mu\vec{J}} \quad \text{--- (7)}$$

From eqn (A), $\vec{\nabla}\cdot\vec{D} = \rho \Rightarrow \vec{\nabla}\cdot\epsilon\vec{E} = \rho$ $\quad \because \vec{D} = \epsilon\vec{E}$

$$\Rightarrow \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon}$$

$$\Rightarrow \vec{\nabla}\cdot\left(-\vec{\nabla}\phi - \frac{\delta\vec{A}}{\delta t}\right) = \frac{\rho}{\epsilon} \text{ using eqn (3) } \vec{E} = -\vec{\nabla}\phi - \frac{\delta\vec{A}}{\delta t}$$

$$\Rightarrow -\vec{\nabla}\cdot(\vec{\nabla}\phi) - \vec{\nabla}\left(\frac{\delta\vec{A}}{\delta t}\right) = \frac{\rho}{\epsilon}$$

$$\Rightarrow -\nabla^2\phi - \text{div}\left(\frac{\delta\vec{A}}{\delta t}\right) = \frac{\rho}{\epsilon} \qquad \because \vec{\nabla}\cdot(\vec{\nabla}\phi) = \nabla^2\phi$$

$$\Rightarrow \nabla^2\phi + \frac{\delta}{\delta t}(\text{div}\vec{A}) = -\frac{\rho}{\epsilon} \quad \text{--- (8)}$$

On subtracting and adding $\mu\epsilon\frac{\delta^2\phi}{\delta t^2}$ in eqn (8), we get

$$\nabla^2\phi - \mu\epsilon\frac{\delta^2\phi}{\delta t^2} + \frac{\delta}{\delta t}(\text{div}\vec{A}) + \mu\epsilon\frac{\delta^2\phi}{\delta t^2} = -\frac{\rho}{\epsilon}$$

$$\Rightarrow \boxed{\nabla^2\phi - \mu\epsilon\frac{\delta^2\phi}{\delta t^2} + \frac{\delta}{\delta t}\left(\text{div}\vec{A} + \mu\epsilon\frac{\delta\phi}{\delta t}\right) = -\frac{\rho}{\epsilon}} \quad \text{--- (9)}$$

Eqns (7) and (9) represent wave equations in terms of scalar potential ϕ and vector potential $\vec{A}$.