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# Relativistic Electrodynamics

## Invariance or Covariance of Maxwell's field equations under Lorentz transformation:

Maxwell's field equation (four equations) are

$$\left. \begin{aligned} \vec{\nabla} \cdot \vec{E} &= \frac{\rho}{\epsilon_0} \text{ --- (i)} \\ \vec{\nabla} \cdot \vec{B} &= 0 \text{ --- (ii)} \\ \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \text{ --- (iii)} \\ \vec{\nabla} \times \vec{B} &= \mu_0 \left( \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \text{ --- (iv)} \end{aligned} \right\} \text{ --- ①}$$

These Maxwell's field equations can be written in terms of electromagnetic potential  $\vec{A}$  and  $\phi$  in two equations only as

$$\square^2 \vec{A} = -\mu_0 \vec{J} \text{ --- ②}$$

$$\square^2 \phi = -\frac{\rho}{\epsilon_0} \text{ --- ③}$$

with the Lorentz condition

$$\text{div } \vec{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0 \text{ --- ④}$$

The electromagnetic four potential (four vector potential)  $A_\mu$  and current four vector  $J_\mu$  are defined as

$$A_\mu = (\vec{A}, \frac{i\phi}{c}) \text{ --- ⑤}$$

$$\text{and } J_\mu = (\vec{J}, ic\beta) \text{ --- ⑥}$$

These four vectors obey the Lorentz transformation.

$$\text{so } A'_\mu = \alpha_{\mu\nu} \cdot A_\nu \text{ --- ⑦}$$

$$\text{and } J'_\mu = \alpha_{\mu\nu} \cdot J_\nu \text{ --- ⑧}$$

where

$$\alpha_{\mu\nu} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

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Lorentz transformation of four vector potential  $A_\mu$  is

$$\left. \begin{aligned} A'_1 &= \gamma(A_1 - \frac{v}{c^2}\phi) , \quad A'_2 = A_2 , \quad A'_3 = A_3 \\ \text{and } A'_4 &= \gamma(A_4 - i\beta A_1) \quad \text{or } \phi' = \gamma(\phi - vA_1) \end{aligned} \right\} \text{--- (9)}$$

Lorentz transformation of current four vector  $J_\mu$  is

$$\left. \begin{aligned} J'_1 &= \gamma(J_1 - v\rho) , \quad J'_2 = J_2 , \quad J'_3 = J_3 \\ \text{and } J'_4 &= \gamma(J_4 - i\beta J_1) \quad \text{or } \rho' = \gamma(\rho - \frac{v}{c^2}J_1) \end{aligned} \right\} \text{--- (10)}$$

In terms of four vector potential  $A_\mu$  and current four vector  $J_\mu$ , Maxwell's field equation can be written in single equation as

$$\square^2 A_\mu = -\mu_0 J_\mu \text{--- (11)}$$

with Lorentz condition  $\frac{\partial A_\mu}{\partial x_\mu} = 0 \text{--- (12)}$

The invariance (or covariance) of Maxwell's field equations require that in any inertial frame  $S'$ , equations (11) and (12) must retain the same form, i.e., Maxwell's field equation will be invariant if

$$\left. \begin{aligned} \square'^2 A'_\mu &= -\mu_0 J'_\mu \\ \text{with Lorentz condition } \frac{\partial A'_\mu}{\partial x'_\mu} &= 0 \end{aligned} \right\} \text{--- (13)}$$

Now (a)  $\square'^2 A'_1 = \square^2 A'_1 \quad \because \square'^2 = \square^2$  D'Alembertian operator is invariant.

$$= \square^2 \left[ \gamma \left( A_1 - \frac{v}{c^2} \phi \right) \right] \text{ using eqn (9)}$$

$$= \gamma \left[ \square^2 A_1 - \frac{v}{c^2} \square^2 \phi \right]$$

$$= \gamma \left[ -\mu_0 J_1 - \frac{v}{c^2} \left( -\frac{\rho}{\epsilon_0} \right) \right]$$

$$= \gamma \left( -\mu_0 J_1 + \mu_0 v \rho \right)$$

$$= -\mu_0 \left[ \gamma (J_1 - v\rho) \right]$$

$$\Rightarrow \square'^2 A'_1 = -\mu_0 J'_1 \text{--- (14) using eqn (10)}$$

$$\because \square^2 A_1 = -\mu_0 J_1$$

$$\square^2 \phi = -\frac{\rho}{\epsilon_0}$$

$$\because c^2 = \frac{1}{\mu_0 \epsilon_0} \Rightarrow \mu_0 = \frac{1}{c^2 \epsilon_0}$$

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$$\begin{aligned}
 (b) \quad \square'^2 A'_2 &= \square^2 A'_2 \quad \because \square'^2 = \square^2 \\
 &= \square^2 A_2 \quad \because A'_2 = A_2 \text{ using eqn } (9) \\
 &= -\mu_0 J_2 \quad \text{Using eqn } (10) \quad \square^2 A_2 = -\mu_0 J_2 \\
 \square'^2 A'_2 &= -\mu_0 J'_2 \quad \text{--- 14 (b)} \quad \because J'_2 = J_2 \text{ using eqn } (10)
 \end{aligned}$$

$$\begin{aligned}
 (c) \quad \square'^2 A'_3 &= \square^2 A'_3 \quad \because \square'^2 = \square^2 \\
 &= \square^2 A_3 \quad \because A'_3 = A_3 \text{ using eqn } (9) \\
 &= -\mu_0 J_3 \quad \because \square^2 A_3 = -\mu_0 J_3 \text{ using eqn } (10) \\
 \square'^2 A'_3 &= -\mu_0 J'_3 \quad \text{--- 14 (c)} \quad \because J'_3 = J_3 \text{ using eqn } (10)
 \end{aligned}$$

$$\begin{aligned}
 (d) \quad \square'^2 A'_4 &= \square^2 A'_4 \quad \because \square'^2 = \square^2 \\
 &= \square^2 [\gamma(A_4 - i\beta A_1)] \quad \because A'_4 = \gamma(A_4 - i\beta A_1) \text{ using eqn } (9) \\
 &= \gamma [\square^2 A_4 - i\beta \square^2 A_1] \\
 &= \gamma [-\mu_0 J_4 - i\beta (-\mu_0 J_1)] \quad \because \square^2 A_\mu = -\mu_0 J_\mu \text{ using eqn } (10) \\
 &= -\mu_0 [\gamma(J_4 - i\beta J_1)] \\
 \square'^2 A'_4 &= -\mu_0 J'_4 \quad \text{--- 14 (d)} \quad \because J'_4 = \gamma(J_4 - i\beta J_1) \text{ using eqn } (10)
 \end{aligned}$$

Eqns 14 (a), 14 (b), 14 (c) and 14 (d) can be jointly written in the form of single equation as

$$\square'^2 A'_\mu = -\mu_0 J'_\mu \quad \text{--- (15)}$$

Further we have

$$A'_\mu = a_{\mu\nu} A_\nu \quad \text{From eqn } (9)$$

$$\Rightarrow A'_\mu = \frac{\partial x'_\mu}{\partial x_\nu} A_\nu \quad \because x'_\mu = a_{\mu\nu} x_\nu \Rightarrow \frac{\partial x'_\mu}{\partial x_\nu} = a_{\mu\nu}$$

$$\Rightarrow \frac{\partial A'_\mu}{\partial x'_\mu} = \frac{\partial}{\partial x'_\mu} \cdot \frac{\partial x'_\mu}{\partial x_\nu} \cdot A_\nu$$

$$\Rightarrow \frac{\partial A'_\mu}{\partial x'_\mu} = \frac{\partial A_\nu}{\partial x_\nu}$$

(4)

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But from Lorentz condition (12)  $\frac{\partial A_{\mu}}{\partial x_{\mu}} = 0$

$$\text{so } \frac{\partial A'_{\mu}}{\partial x'_{\mu}} = 0 \quad (16)$$

Thus equations (15) and (16) are same as equation (13).

Therefore equations (11) and (12) retain the same form when expressed in terms of primed quantities.

So Maxwell's field equations are invariant under Lorentz transformation.

Ques:- To prove that Maxwell's field equations are invariant (or covariant) under Lorentz transformation.