Reduced to Normal Form (Removal of first derivative)

Method 2. To Find the Complete Solution of $y''+Py'+Qy=R$ **when it is reduced to Normal Form (Removal of first derivative)**

When the part of C.F. can not be determined by the previous method, we reduce the given differential equation in **normal form** by eliminating the term in which there exists first derivative of the dependent variable.

$$\frac{d^2y}{dx^2}P\frac{dy}{dx}+Qy=R \qquad \ldots(1)$$

Let $y=uv$ be the complete solution of eqn. (1), where u and v are the function of x.

$\therefore$ $$\frac{dy}{dx}=v\frac{du}{dx}+u\frac{dv}{dx}$$

and $$\frac{d^2y}{dx^2}=v\frac{d^2u}{dx^2}+2\frac{du}{dx}\frac{dv}{dx}+u\frac{d^2v}{dx^2}$$

Substituting the value of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in eqn. (1), we get

$$\frac{d^2v}{dx^2}+\left(\frac{2}{u}\frac{du}{dx}+P\right)\frac{du}{dx}+v\left(\frac{1}{u}\frac{d^2u}{dx^2}+\frac{P}{u}\frac{du}{dx}+Q\right)=\frac{R}{u}$$

…(2)

Let us choose u such that $\frac{2}{u}\frac{du}{dx}+P=0$

…(3)

Which on solving gives,

$$u=e^{-\int\frac{P}{2}dx}$$

…(4)

From (3), $$\frac{du}{dx}=-\frac{Pu}{2}$$

Differentiating, we get $\frac{d^2u}{dx^2}=-\frac{1}{2}\left[P\left(\frac{du}{dx}\right)+\frac{dP}{dx}(u)\right]$

$$= -\frac{1}{2}\left[P\left(\frac{-Pu}{2}\right) + u\frac{dP}{dx}\right] = \frac{P^2 u}{2} - \frac{u}{2}\frac{dP}{dx}$$

Coefficient of $v = \frac{1}{u}\frac{d^2u}{dx^2} + \frac{P}{u}\frac{du}{dx} + Q = \frac{1}{u}\left[\frac{P^2 u}{4} - \frac{u}{2}\frac{dP}{dx}\right] + \frac{P}{u}\left(\frac{-Pu}{2}\right) + Q$

$$= Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4} = I \text{ (say)}$$

Then (2) becomes, $\frac{d^2v}{dx^2} + Iv = S$ …(5)

This is known as the normal form of equation (1).

Solving (5), we get v in terms of x. Ultimately, $y = u\upsilon$ is the complete solution.

Q.24 Solve : $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 1)y = -3e^{x^2}\sin 2x.$

Sol. Here, $P = -4x, Q = 4x^2 - 1, R = -3e^{x^2}\sin 2x$

Let $y = u\upsilon$ be the complete solution.

Now, $u = e^{-\frac{1}{2}\int(-4x)dx} = e^{x^2}$

$$I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4} = 4x^2 - 1\frac{1}{2}(-4) - \frac{1}{4}(16x^2) = 1.$$

Also, $S = \frac{R}{u} = \frac{-3e^{x^2}\sin 2x}{e^{x^2}} = -3\sin 2x$

Hence normal form is,

$$\frac{d^2\upsilon}{dx^2} + \upsilon = -3\sin 2x$$

Auxiliary equation is

$$m^2 + 1 = 0 \;\Rightarrow\; m = \pm i$$

$$C.F. = c_1\cos x + c_2\sin x$$

where c_1 and c_2 are arbitrary constants of integration.

$$P.I. = \frac{1}{D^2+1}(-3\sin 2x) = \frac{-3}{(-4+1)}\sin 2x = \sin 2x$$

$\therefore$ Solution is, $\upsilon = c_1 \cos x + c_2 \sin x + \sin 2x$

Hence the complete solution of given differential equation is

$$y = u\upsilon = e^{x^2}(c_1 \cos x + c_2 \sin x + \sin 2x)\,.$$

1.10- Changing the Independent Variable

Method 3. To Find the Complete Solution of $y'' + Py' + Qy = R$ Changing the Independent Variable

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad \ldots(1)$$

Let us relate x and z by the relation,

$$z = f(x) \qquad \ldots(2)$$

$$\therefore \qquad \frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} \qquad \ldots(3)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dz}\cdot\frac{dz}{dx}\right)$$

$$= \frac{dy}{dz}\cdot\frac{d^2z}{dx^2} + \frac{dz}{dx}\cdot\frac{d}{dz}\left(\frac{dy}{dz}\right)\cdot\frac{dz}{dx} = \frac{dy}{dz}\cdot\frac{d^2z}{dx^2} + \left(\frac{dz}{dz}\right)^2\frac{d^2y}{dx^2} \qquad \ldots(4)$$

Substituting in (1), we get

$$\frac{dy}{dz}\cdot\frac{d^2z}{dx^2} + \left(\frac{dz}{dx}\right)^2\frac{d^2y}{dx^2} + P\frac{dy}{dz}\cdot\frac{dz}{dx} + Qy = R$$

$$\Rightarrow \qquad \frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1 \qquad \ldots(5)$$

where $P_1 = \dfrac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2}, Q_1 = \dfrac{Q}{\left(\dfrac{dz}{dx}\right)^2}, R_1 = \dfrac{R}{\left(\dfrac{dz}{dx}\right)^2}$

Here P_1, Q_1, R_1 are functions of x which can be transformed into functions of z using the relation $z = f(x)$.

Choose z such that $Q_1 = \text{constant} = a^2$ (say)

$$\Rightarrow \qquad \frac{Q}{\left(\frac{dz}{dx}\right)^2} a^2 \qquad \Rightarrow \qquad a\frac{dz}{dx} = \sqrt{Q}$$

$$\Rightarrow \qquad dz = \frac{\sqrt{Q}}{a}dx$$

Integration yields, $\qquad z = \int \frac{\sqrt{Q}}{a}dx$

If this value of z makes P_1 as constant then equation (5) can be solved.

Q.25 Solve : $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos\log(1+x)$.

Sol. $\qquad \frac{d^2y}{dx^2} + \frac{1}{1+x}\frac{dy}{dx} + \frac{y}{(1+x)^2} = \frac{4}{(1+x)^2}\cos\log(1+x)$

…(1)

Choose z such that,

$$\left(\frac{dz}{dx}\right)^2 = \frac{1}{(1+x)^2}$$

$$\Rightarrow \qquad \frac{dz}{dx} = \frac{1}{1+x}$$

…(2)

Integration yields, $\qquad z = \log(1+x)$

…(3)

From(2), $\qquad \frac{d^2z}{dx^2} = -\frac{1}{(1+x)^2}$

$$\therefore \qquad P_1 = \frac{-\frac{1}{(1+x)^2} + \frac{1}{1+x}.\frac{1}{1+x}}{\frac{1}{(1+x)^2}} = 0$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = 1$$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = 4\cos\log(1+x) = 4\cos z \qquad |\text{Form (3)}$$

Reduced equation is

$$\frac{d^2y}{dz^2} + y = 4\cos z$$

Auxiliary equation is $m^2 + 1 = 0 \qquad \Rightarrow m = \pm i$

$$C.F. = c_1\cos z + c_2\sin z$$

$$P.I. = \frac{1}{D^2+1}(4\cos z) = 4.\frac{z}{2}\sin z = 2z\sin z$$

Complete solution is

$$y = c_1\cos z + c_2\sin z + 2z\sin z$$

$$y = c_1\cos\log(1+x) + c_2\sin\log(1+x) + 2\log(1+x)\sin\log(1+x).$$